

Computational Physics

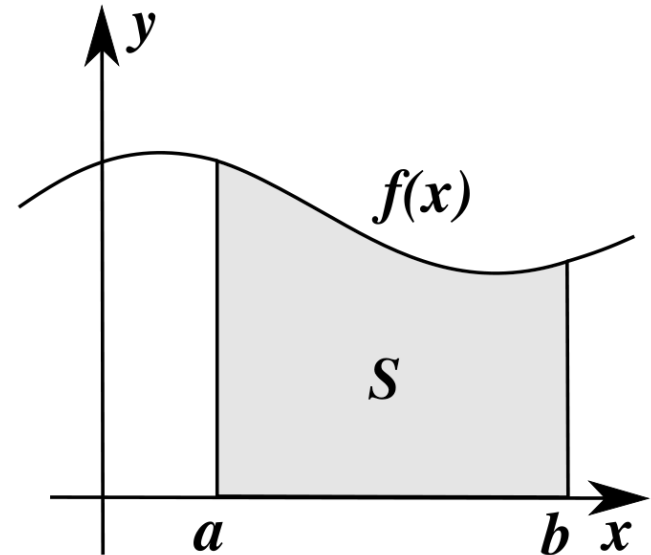
Numerical Integration

- Rectangular rule
- Trapezoidal rule
- Simpson rule
- Newton-Cotes rules
- Gauss-Legendre quadrature

(Numerical) Quadrature

- In general, a numerical integration is the approximation of a definite integration by a “weighted” sum of function values at discretized points within the interval of integration.

$$\int_a^b f(x) dx \approx \sum_{i=0}^N w_i f(x_i)$$



where w_i is the weight depending on the integration schemes used, and $f(x_i)$ is the function value evaluated at the given point $x_i \in [a, b]$

Methods

- Rectangular rule
 - Lower and Upper sums
 - Midpoint sums
- Trapezoidal rule
- Simpson's rule
- Newton-Cotes rules
- Gauss-Legendre quadrature

- (Monte-Carlo integration)

Riemann integrals

Consider a smooth function $f(x)$ in $[a,b]$, then the Riemann integral is defined as:

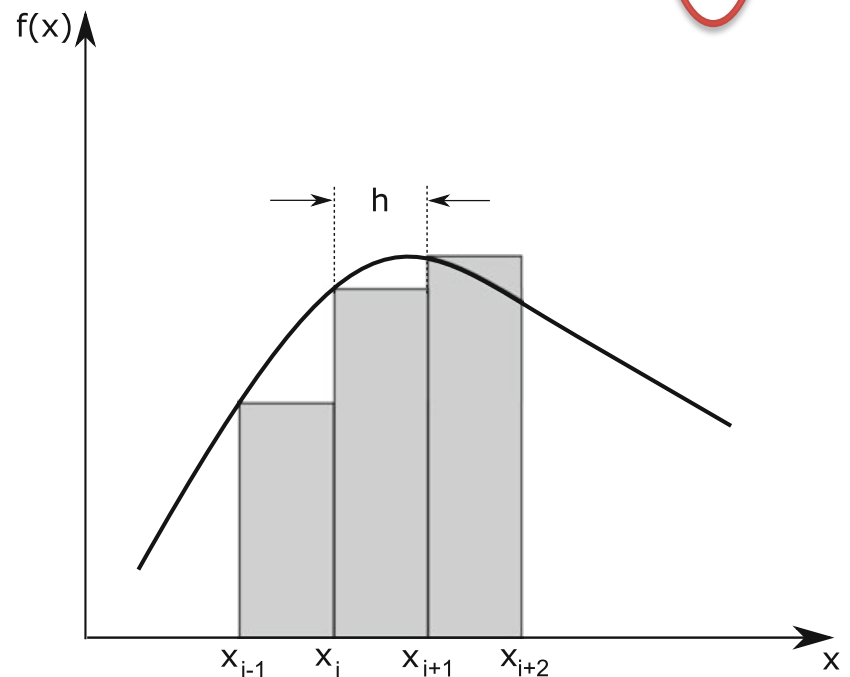
$$\int_a^b dx f(x) = \lim_{N \rightarrow \infty} \frac{b-a}{N} \sum_{i=0}^N f\left(a + i \frac{b-a}{N}\right)$$

Discretization of $[a,b]$ in equally space grid-points x_i yields $\int_a^b dx f(x) \approx h \sum_{i=1}^{N-1} f_i$

Note
$$\int_a^b dx f(x) = \sum_{i=1}^{N-1} \int_{x_i}^{x_{i+1}} dx f(x)$$

and therefore for an elemental area

$$\int_{x_i}^{x_{i+1}} dx f(x) \approx h f_i$$



Error of discretization

The error of this approximation can be found by expanding $f(x)$ in a Taylor series

$$\begin{aligned}\int_{x_i}^{x_{i+1}} dx f(x) &= \int_{x_i}^{x_{i+1}} dx \left[f_i + (x - x_i) f'_{i+\varepsilon_\xi} \right] \\ &= f_i h + \mathcal{O}(h^2).\end{aligned}$$

Where we used the mean value theorem for a continuous function $f(x)$ in $[a,b]$:

$$\int_a^b dx f(x) = (b - a) f(\zeta) \quad \text{for some } \zeta \text{ in } [a,b]$$

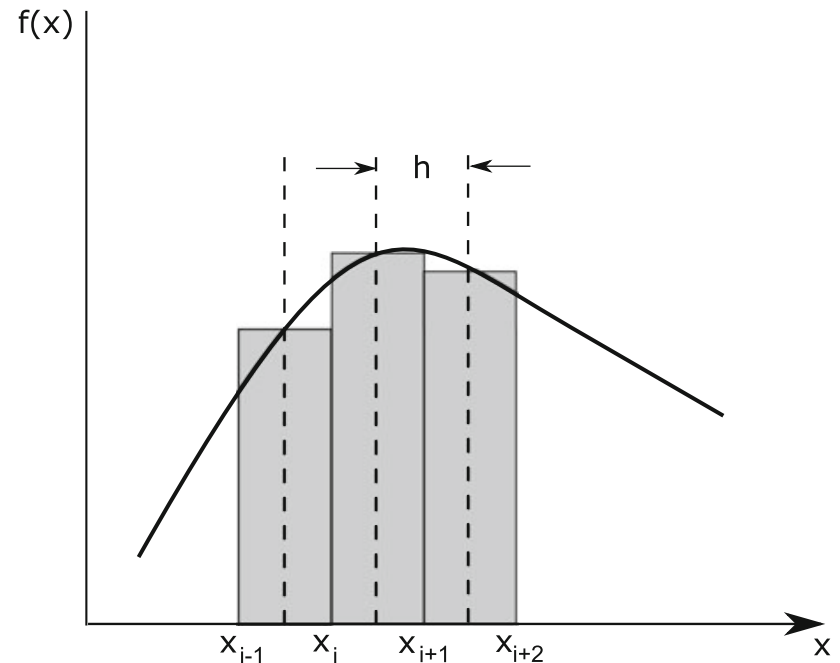
The procedure above corresponds to the forward difference approach, similarly for backward:

$$\int_a^b dx f(x) = h \sum_{i=2}^N f_i + \mathcal{O}(h^2)$$

Central difference: Rectangular rule

Accordingly, we get the following expression for the central difference, which has higher accuracy:

$$\begin{aligned} \int_{x_i}^{x_{i+1}} dx f(x) &= \int_{x_i}^{x_{i+1}} dx \left\{ f_{i+\frac{1}{2}} + \left(x - x_{i+\frac{1}{2}}\right) f'_{i+\frac{1}{2}} \right. \\ &\quad \left. + \frac{\left(x - x_{i+\frac{1}{2}}\right)^2}{2} f''_{i+\frac{1}{2}} + \mathcal{O}\left[\left(x - x_{i+\frac{1}{2}}\right)^3\right] \right\} \\ &= hf_{i+\frac{1}{2}} + \frac{h^3}{24} f''_{i+\frac{1}{2}} + \mathcal{O}(h^5) \end{aligned}$$



I.e. the error is of order h^3 (instead of h^2)

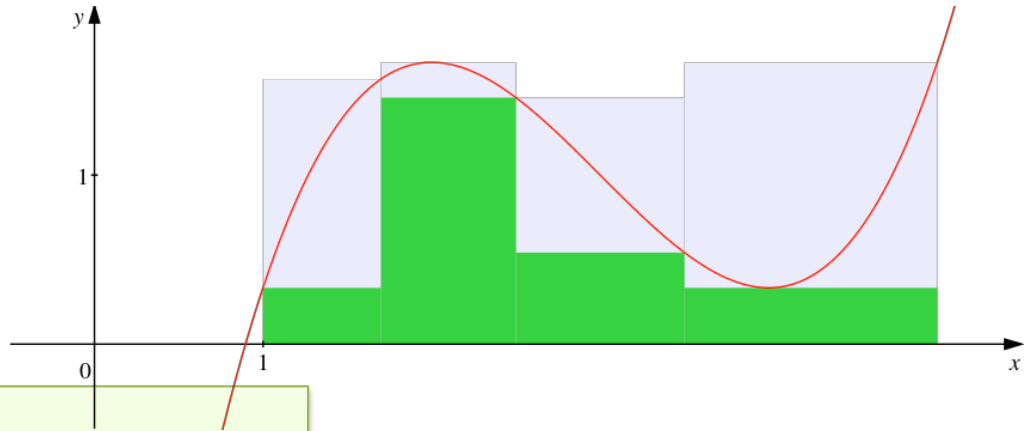
$$\int_a^b dx f(x) = h \sum_{i=1}^{N-1} f_{i+\frac{1}{2}} + \mathcal{O}(h^3) \quad \text{Rectangular rule or Midpoint rule}$$

Note, the boundary points a and b of the integration interval do not enter:

Example of an *open* integration rule. If the end-points are used the method is called *closed*.

Lower and Upper Sums

The lower and upper sums are defined as:



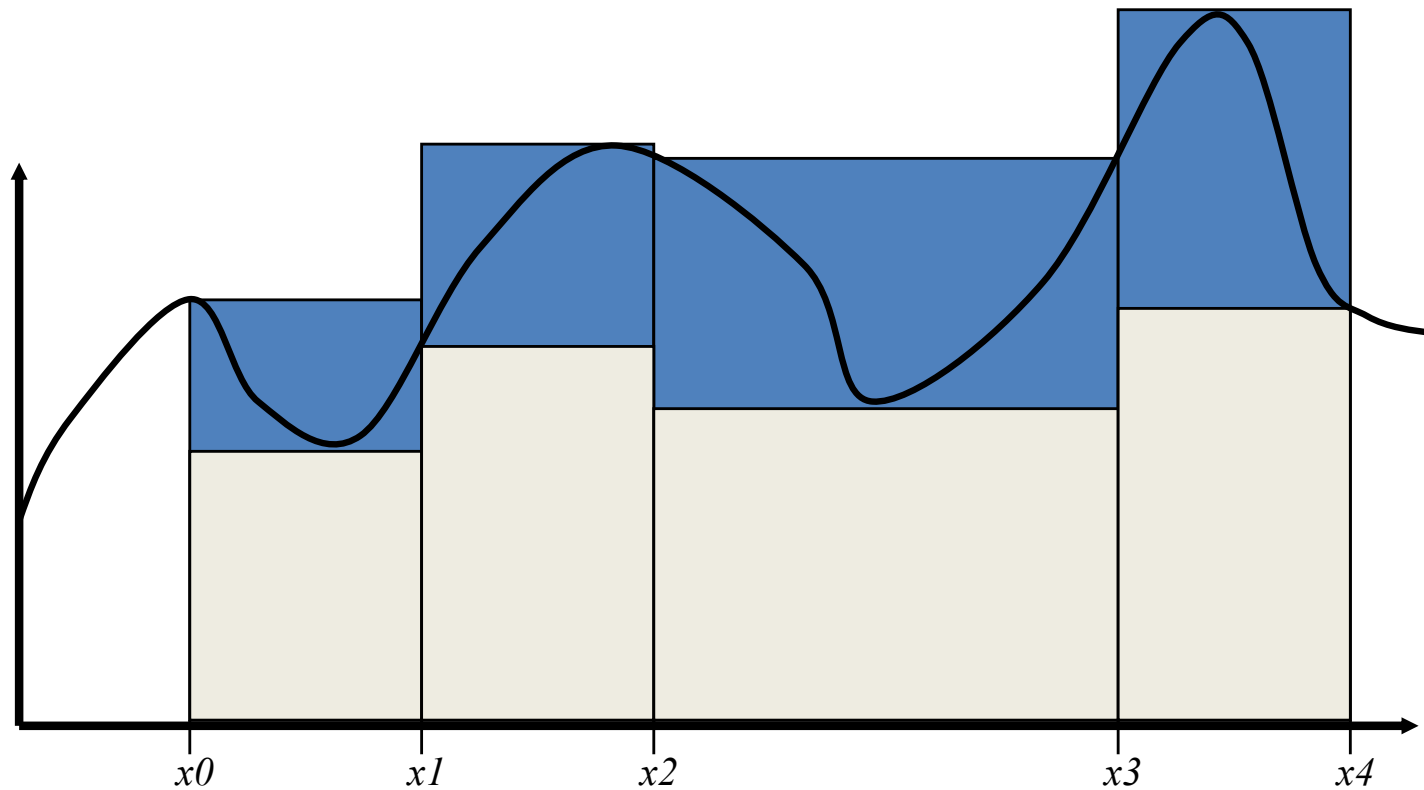
$$\begin{aligned} \text{Lower: } \Sigma^- &\equiv \sum_{k=1}^N (x_k - x_{k-1}) \cdot \inf_{x_{k-1} < x < x_k} f(x) \\ \text{Upper: } \Sigma^+ &\equiv \sum_{k=1}^N (x_k - x_{k-1}) \cdot \sup_{x_{k-1} < x < x_k} f(x) \end{aligned}$$

Define lower and upper bounds for the real integral, but these are impractical because of “inf” and “sup”.

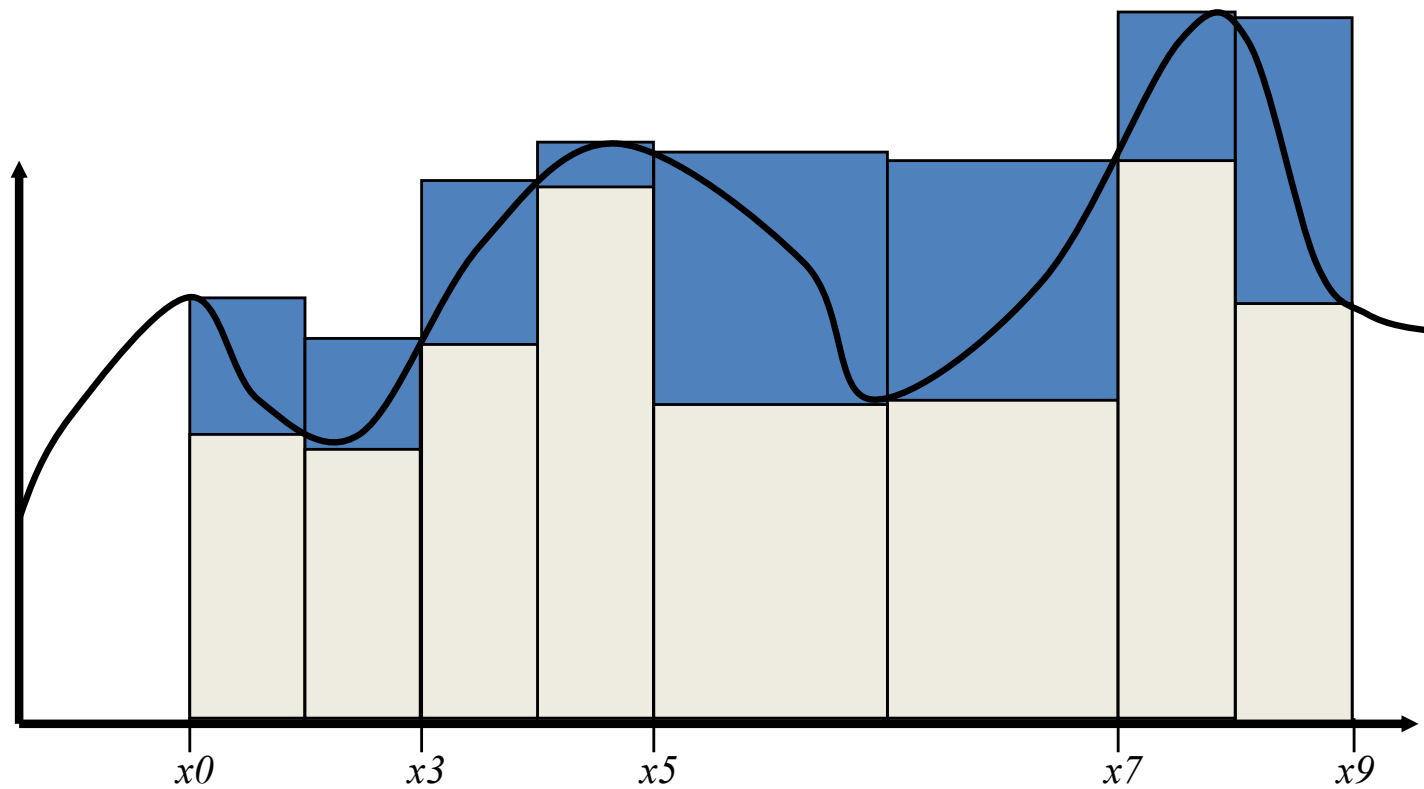
More practical is the rectangular rule.

Bounding approximations

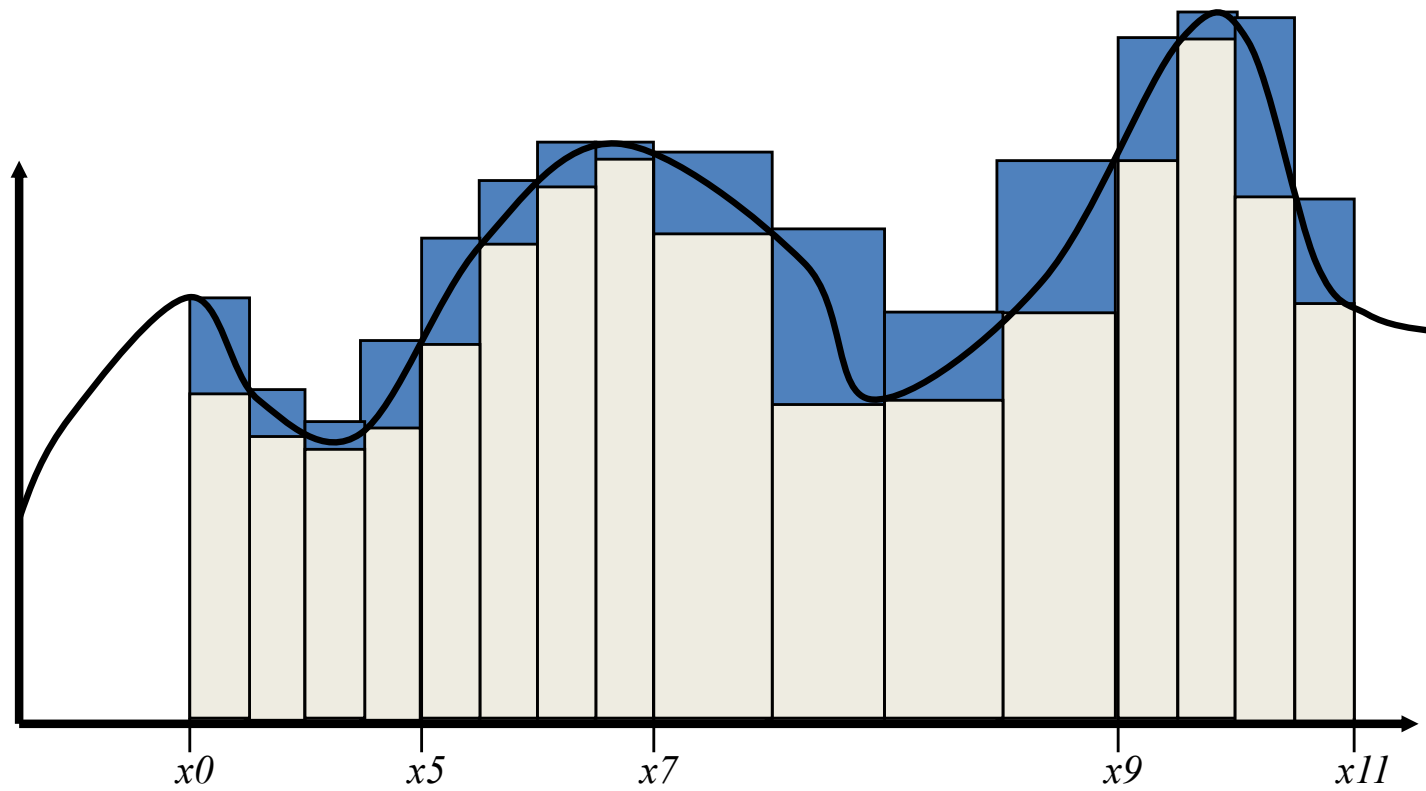
$$\Sigma^- \leq \int_a^b f(x) dx \leq \Sigma^+$$



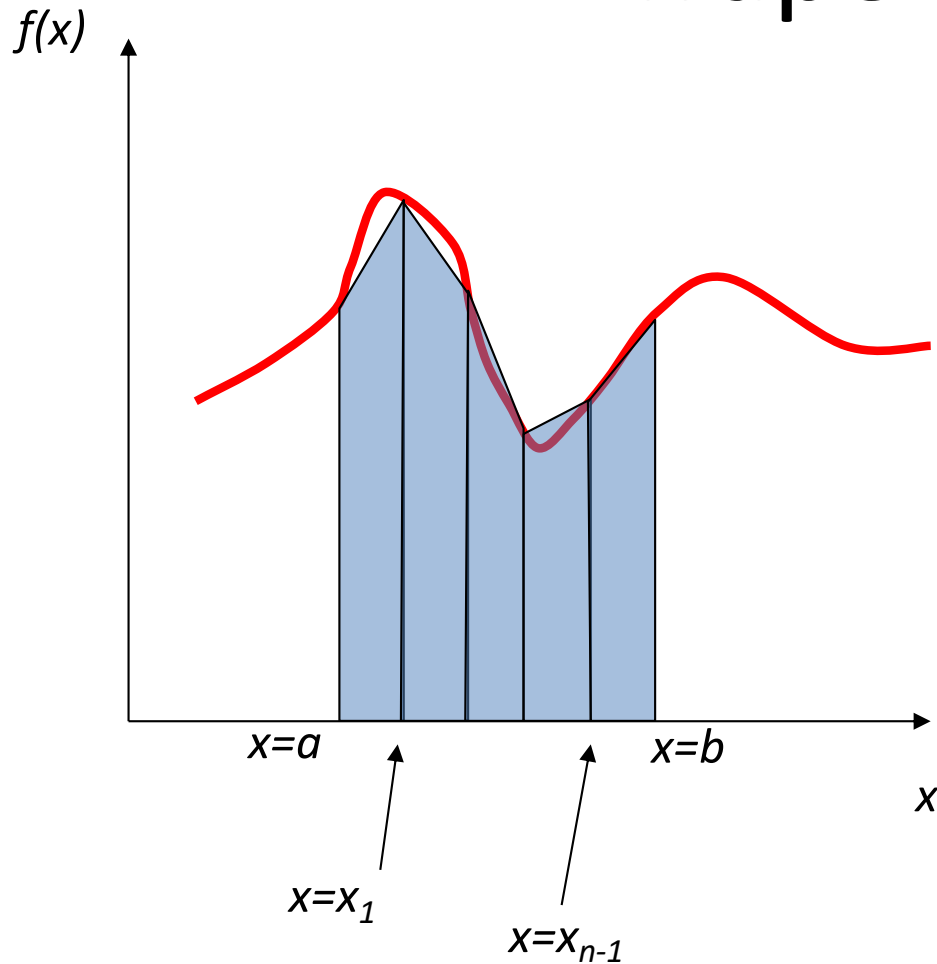
Can be refined



Can be refined



Trapezoidal rule



- Needs function values only at x_i
- Error is, like rectangular rule, of order h^3

The rectangular rule can be made more accurate by using trapezoids to replace the rectangles as shown. A linear approximation of the function locally sometimes work much better than using the averaged value like the rectangular rule does.

Elemental area $\int_{x_i}^{x_{i+1}} dx f(x) \approx \frac{h}{2} (f_i + f_{i+1})$

Trapezoidal rule: (closed)

$$\begin{aligned} \int_a^b dx f(x) &\approx \frac{h}{2} \sum_{i=1}^{N-1} (f_i + f_{i+1}) \\ &= h \left(\frac{f_1}{2} + f_2 + \dots + f_{N-1} + \frac{f_N}{2} \right) \\ &= \frac{h}{2} (f_1 + f_N) + h \sum_{i=2}^{N-1} f_i \end{aligned}$$

Simpson (1/3) rule

As we saw for the finite difference approximation of derivatives, the accuracy can be improved by using more grid points

There we expand $f(x)$ around the midpoint x_i of integration interval $[x_{i-1}, x_{i+1}]$

$$\begin{aligned}\int_{x_{i-1}}^{x_{i+1}} dx f(x) &= \int_{x_{i-1}}^{x_{i+1}} dx \left[f_i + (x - x_i) f'_i + \frac{(x - x_i)^2}{2!} f''_i \right. \\ &\quad \left. + \frac{(x - x_i)^3}{3!} f'''_i + \dots \right] \\ &= 2hf_i + \frac{h^3}{3} f''_i + \mathcal{O}(h^5).\end{aligned}$$

For the second derivate at x_i we use $f''_i \approx (f_{i+1} - 2f_i + f_{i-1})/h^2$

$$\begin{aligned}\int_{x_{i-1}}^{x_{i+1}} dx f(x) &= 2hf_i + \frac{h}{3} (f_{i+1} - 2f_i + f_{i-1}) + \mathcal{O}(h^5) \\ &= h \left(\frac{1}{3} f_{i-1} + \frac{4}{3} f_i + \frac{1}{3} f_{i+1} \right) + \mathcal{O}(h^5).\end{aligned}$$

Simpson rule

This is a three point rule and has error h^5 ! (lucky cancellation of third derivate)

...

The composite form for interval $[a,b]$ with even N follows from:

$$\int_a^b dx f(x) = \int_{x_0}^{x_2} dx f(x) + \int_{x_2}^{x_4} dx f(x) + \dots + \int_{x_{N-2}}^{x_N} dx f(x)$$

we get ($x_i = a + ih$)

$$\int_a^b dx f(x) = \frac{h}{3} (f_0 + 4f_1 + 2f_2 + 4f_3 + \dots + 2f_{N-2} + 4f_{N-1} + f_N) + \mathcal{O}(h^5).$$

This expression is exact for polynomials of degree $n \leq 3$ since the first term in the error expansion involves the fourth derivative.

If the integrand is satisfactorily reproduceable by a polynomial of degree three or less, the Simpson rule gives almost exact estimates, independent of h .

Example: Simpson's rule for x^3

$$\begin{aligned}\int_a^b x^3 dx &= \frac{1}{4}(b^4 - a^4) = \frac{1}{4}(b-a)(b^3 + b^2a + ba^2 + a^3) \\&= \frac{1}{4}(b-a)\left(\frac{2}{3}b^3 + \frac{1}{3}b^3 + b^2a + ba^2 + \frac{1}{3}a^3 + \frac{2}{3}a^3\right) \\&= \frac{1}{4}(b-a)\left(\frac{2}{3}b^3 + \frac{1}{3}(b+a)^3 + \frac{2}{3}a^3\right) \\&= \frac{1}{4}(b-a)\left(\frac{2}{3}b^3 + \frac{8}{3}\left(\frac{b+a}{2}\right)^3 + \frac{2}{3}a^3\right) \\&= \frac{1}{6}(b-a)\left(b^3 + 4\left(\frac{b+a}{2}\right)^3 + a^3\right) \\&= \frac{1}{6}(b-a)\left(f(a) + 4f\left(\frac{b+a}{2}\right) + f(b)\right)\end{aligned}$$

Higher-order approximations

For four-point approximations we get the Simpson (3/8) rule:

$$\int_{x_i}^{x_{i+3}} dx f(x) = \frac{3h}{8} (f_i + 3f_{i+1} + 3f_{i+2} + f_{i+3}) + \mathcal{O}(h^5)$$

five points:
Bode's rule: $\int_{x_1}^{x_5} f(x) dx = h \left[\frac{14}{45} f_1 + \frac{64}{45} f_2 + \frac{24}{45} f_3 + \frac{64}{45} f_4 + \frac{14}{45} f_5 \right] + \mathcal{O}(h^7 f^{(6)})$

Remarks:

- Simpson's 1/3 rule is exact for polynomials up to order 3
- Simpson's 3/8 rule is also exact for polynomials up to order 3!
(no "lucky cancellation")
- Bode's rule is exact for polynomials up to order 5!

General: Newton-Cotes rules

The previous approximations can be formalized as follows:

Define the Lagrange interpolation polynomial $p_{n-1}(x)$ of degree $n-1$ to a function $f(x)$:

$$p_{n-1}(x) = \sum_{j=1}^n f_j L_j^{(n-1)}(x) \quad \text{with} \quad L_j^{(n-1)}(x) = \prod_{\substack{k=1 \\ k \neq j}}^n \frac{x - x_k}{x_j - x_k}$$

where $p_{n-1}(x_j) = f_j$

With this, any smooth function $f(x)$ can be written as

$$f(x) = p_{n-1}(x) + \frac{f^{(n)}[\xi(x)]}{n!} (x - x_1)(x - x_2) \dots (x - x_n)$$

Neglecting the second term and integrating over the n grid-points the Lagrange polynomial only, one obtains the closed n -point Newton-Cotes formulas.

One can also define open Newton-Cotes rule ($n=1$ gives the rectangular rule), but these are typically inferior to Gauss-Legendre quadratures (which are open). *see textbook*

Example: n=2

For n=2, the Lagrange polynomial $p_1(x)$ to $f(x)$ is

$$\begin{aligned} p_1(x) &= f_1 L_1^{(1)}(x) + f_2 L_2^{(1)}(x) \\ &= f_1 \frac{x - x_2}{x_1 - x_2} + f_2 \frac{x - x_1}{x_2 - x_1} \\ &= \frac{1}{h} [x(f_2 - f_1) - x_1 f_2 + x_2 f_1] \end{aligned}$$

And therefore

$$\begin{aligned} \int_{x_1}^{x_2} dx p_1(x) &= \frac{1}{h} \left[\frac{x^2}{2} (f_2 - f_1) + x(x_2 f_1 - x_1 f_2) \right] \Big|_{x_1}^{x_2} \\ &= \frac{h}{2} [f_2 + f_1] , \end{aligned}$$

which is the trapezoidal rule.

For n=3 one get Simpson's (1/3) rule,

for n=4 Simpson's 3/8 rule, and for n=5 Bode's rule

Adaptive algorithms

Much better than to predefine the number of sub-intervals, N , is to refine the integration rule until some specified degree of accuracy has been achieved.

Example:

adaptive
refinement of
trapezoidal rule
(equivalent to
Simpson's 1/3 rule)

```
#include <math.h>
#define EPS 1.0e-6
#define JMAX 20

float qsimp(float (*func)(float), float a, float b)
Returns the integral of the function func from a to b. The parameters EPS can be set to the
desired fractional accuracy and JMAX so that 2 to the power JMAX-1 is the maximum allowed
number of steps. Integration is performed by Simpson's rule.
{
    float trapzd(float (*func)(float), float a, float b, int n);
    void nrerror(char error_text[]);
    int j;
    float s,st,ost=0.0,os=0.0;

    for (j=1;j<=JMAX;j++) {
        st=trapzd(func,a,b,j);
        s=(4.0*st-ost)/3.0;
        if (j > 5)
            if (fabs(s-os) < EPS*fabs(os) ||
                (s == 0.0 && os == 0.0)) return s;
        os=s;
        ost=st;
    }
    nrerror("Too many steps in routine qsimp");
    return 0.0;
}
```

Compare equation (4.2.4), above.
Avoid spurious early convergence.
Never get here.

Legendre polynomials

Orthogonal Legendre polynomials $P_l(x)$ are solutions of the Legendre differential equation:

$$(1 - x^2) P_\ell''(x) - 2xP_\ell'(x) + \ell(\ell + 1)P_\ell(x) = 0$$

and are given by $P_\ell(x) = \sum_{k=0}^{\infty} a_{k,\ell} x^k$ with $a_{k+2,\ell} = \frac{k(k+1) - \ell(\ell+1)}{(k+1)(k+2)} a_{k,\ell}$

Properties

- for even l , $P_l(x)$ is even, and odd for odd l ,
- for $k \geq l$, the coefficients are zero
- $P_l(x)$ is of degree l
- fulfills orthonormality:

$$\int_{-1}^1 dx P_\ell(x) P_{\ell'}(x) = \frac{2}{2\ell' + 1} \delta_{\ell\ell'}$$

- $P_0(x)=1$, $P_1(x)=x$
- can be written as (Rodrigues' formula): $P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell} (x^2 - 1)^\ell$

Gauss-Legendre Quadrature

Define the linear transformation of $f(x)$: $F(x) = \frac{b-a}{2}f\left(\frac{b-a}{2}x + \frac{b+a}{2}\right)$

which gives: $\int_a^b dx f(x) = \int_{-1}^1 dx F(x)$

Now we use polynomial approximation (if possible): $F(x) \approx p_{2n-1}(x)$

- $p_{2n-1}(x)$ is some polynomial of degree $2n-1$
- the error is proportional to $F^{(2n)}(x)$.
- If $p_{2n-1}(x)$ is explicitly given, we can use Newton-Cotes formulas
- otherwise, we write:

$$\int_{-1}^1 dx F(x) = \sum_{i=1}^n \omega_i F(x_i)$$

with some weights ω_i and grid-points x_i , which are undetermined!

Weights and grid points

Task:

find weights ω_i and grid-points x_i such that the integral is best approximated.

- start with the decomposition: $p_{2n-1}(x) = p_{n-1}(x)P_n(x) + q_{n-1}(x)$

where p&q are a polynomials of degree n-1, $P_n(x)$ the Legendre polynomial of degree n

Since p is a polynomial, we can also write: $p_{n-1}(x) = \sum_{i=0}^{n-1} a_i P_i(x)$

Using orthonormality we get:

$$\int_{-1}^1 dx p_{2n-1}(x) = \sum_{i=0}^{n-1} a_i \int_{-1}^1 dx P_i(x) P_n(x) + \int_{-1}^1 dx q_{n-1}(x) = \int_{-1}^1 dx q_{n-1}(x) .$$

Since $P_n(x)$ is a Legendre polynomial, it has n zeros in the interval $[-1,1]$, such that

$$p_{2n-1}(x_i) = q_{n-1}(x_i) \quad \text{where } x_i \text{ are the zeros of } P_n(x), \text{ which will be our grid points}$$

Note, these zeros are independent of $F(x)$ and therefore $p_{2n-1}(x)$!

...

Expanding also q in terms of Legendre polynomials:
we get:

$$q_{n-1}(x) = \sum_{i=0}^{n-1} b_i P_i(x)$$

$$p_{2n-1}(x_i) = \sum_{k=0}^{n-1} b_k P_k(x_i), \quad i = 1, \dots, n$$

$$p_i = \sum_{k=0}^{n-1} b_k P_{ki}$$

and $p_i \approx F_i$

inverse matrix
of $\mathbf{P} = \{P_{ik}\}$ which
is non-singular



linear equation system for b_k :

$$b_k = \sum_{i=1}^n F_i [\mathbf{P}^{-1}]_{ik}$$

then we can write:
$$\int_{-1}^1 dx F(x) \approx \int_{-1}^1 dx p_{2n-1}(x) = \sum_{k=0}^{n-1} b_k \int_{-1}^1 dx P_k(x)$$

and use trick $[P_0(x)=1]$:
$$\int_{-1}^1 dx P_k(x) = \int_{-1}^1 dx P_k(x) P_0(x) = \frac{2}{2k+1} \delta_{k0} = 2\delta_{k0}$$

...

which gives us
$$\int_{-1}^1 dx F(x) \approx 2b_0 = 2 \sum_{i=1}^n F_i [\mathbf{P}^{-1}]_{i0}$$

defining
$$\omega_i = 2 [\mathbf{P}^{-1}]_{i0}$$
 one can show that
$$\omega_i = \frac{2}{(1 - x_i^2) [P'_n(x_i)]^2}$$

get finally get:
$$\int_{-1}^1 dx F(x) \approx \sum_{i=1}^n \omega_i F_i$$

Notes:

- the Gauss-Legendre quadrature is exact for polynomials of degree $2n-1$!
- x_i and ω_i are **independent** of $F(x)$ and can be tabled
- the grid-points x_i are symmetrically distributed around 0 and have the same weights
- the Gauss-Legendre quadrature is an open method: ± 1 are not grid-points
- higher accuracy and convergence than comparable Newton-Cotes

Example of deriving GQ

- Here we concentrate on $N=1$ (composite rules are straight forward)

- For $n=2$, we have:
$$I \approx c_1 f(x_1) + c_2 f(x_2)$$

- This leads to 4 unknowns: c_1 , c_2 , x_1 , and x_2
 - two unknown weights (c_1 , c_2)
 - two unknown sampling points (x_1 , x_2)
- *we need four known values for the equation.*
- *If we had these, we could then attempt to solve for the four unknowns.*
- *Let's with polynomials*
- *For $n=2$, let's look at: $1, x, x^2, x^3$*

...

- Recalling the formula: $I \approx c_1 f(x_1) + c_2 f(x_2)$

- Constant
 - $f(x)=1$ $\int_{-1}^1 1 dx = 2 = c_1 f(x_1) + c_2 f(x_2) = c_1 + c_2$
- Linear
 - $f(x)=x$ $\int_{-1}^1 x dx = 0 = c_1 f(x_1) + c_2 f(x_2) = c_1 x_1 + c_2 x_2$
- Quadratic
 - $f(x)=x^2$ $\int_{-1}^1 x^2 dx = \frac{2}{3} = c_1 f(x_1) + c_2 f(x_2) = c_1 x_1^2 + c_2 x_2^2$
- Cubic
 - $f(x)=x^3$ $\int_{-1}^1 x^3 dx = 0 = c_1 f(x_1) + c_2 f(x_2) = c_1 x_1^3 + c_2 x_2^3$

2nd order GQ points/weights

Solving these non-linear equations gives:

$$\begin{aligned}c_1 &= c_2 = 1 \\x_1 &= -\frac{1}{\sqrt{3}} = -0.577 \\x_2 &= \frac{1}{\sqrt{3}} = 0.577\end{aligned}$$

→ Gauss-Legendre formula:

$$I \approx f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

→ This is **exact** for **all** polynomials up to and including degree 3!

3rd order GQ

$$\int_{-1}^1 1 dx = 2 = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 + c_2 + c_3$$

$$\int_{-1}^1 x dx = 0 = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1 + c_2 x_2 + c_3 x_3$$

$$\int_{-1}^1 x^2 dx = \frac{2}{3} = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1^2 + c_2 x_2^2 + c_3 x_3^2$$

$$\int_{-1}^1 x^3 dx = 0 = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1^3 + c_2 x_2^3 + c_3 x_3^3$$

$$\int_{-1}^1 x^4 dx = \frac{2}{5} = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1^4 + c_2 x_2^4 + c_3 x_3^4$$

$$\int_{-1}^1 x^5 dx = 0 = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3) = c_1 x_1^5 + c_2 x_2^5 + c_3 x_3^5$$

...

- Solution of these equations gives

$$\begin{array}{lll} c_1 = 5/9 & c_2 = 8/9 & c_3 = 5/9 \\ x_1 = -\sqrt{3/5} = -0.77459669 & x_2 = 0 & x_3 = \sqrt{3/5} = 0.77459669 \end{array}$$

- Produces the three-point Gauss-Legendre formula

$$I \approx c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

$$I \approx \frac{5}{9} f\left(-\sqrt{\frac{3}{5}}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{\frac{3}{5}}\right)$$

- Exact for polynomials up to and including **degree 5**

GQ weights and Gauss points

n	2	3	4	5	6
c_i	1.0	0.5555555556	0.3478548451	0.2369268850	0.1713245
	1.0	0.8888888889	0.6521451549	0.4786286705	0.3607616
		0.5555555556	0.6521451549	0.5688888889	0.4679139
			0.3478548451	0.4786286705	0.4679139
				0.2369268850	0.3607616
					0.1713245
	-0.5773502692	-0.7745966692	-0.8611363116	-0.9061798459	-0.932469514
x_i	0.5773502692	0.0000000000	-0.3399810436	-0.5384693101	-0.661209386
		0.7745966692	0.3399810436	0.0000000000	-0.238619186
			0.8611363116	0.5384693101	0.238619186
				0.9061798459	0.661209386
					0.932469514

.. even more

TABLE 5.5 Gaussian Quadrature Nodes and Weights

n	$x_i^{(n)}$	$w_i^{(n)}$
1	0.0000000000000000E+00	0.2000000000000000E+01
2	$\pm 0.5773502691896257E+00$	1.0000000000000000E+00
4	$\pm 0.3399810435848563E+00$ $\pm 0.8611363115940526E+00$	0.6521451548625464E+00 0.3478548451374476E+00
8	$\pm 0.1834346424956498E+00$ $\pm 0.5255324099163290E+00$ $\pm 0.7966664774136268E+00$ $\pm 0.9602898564975362E+00$	0.3626837833783620E+00 0.3137066458778874E+00 0.2223810344533745E+00 0.1012285362903697E+00
16	$\pm 0.9501250983763744E-01$ $\pm 0.2816035507792589E+00$ $\pm 0.4580167776572274E+00$ $\pm 0.6178762444026438E+00$ $\pm 0.7554044083550030E+00$ $\pm 0.8656312023878318E+00$ $\pm 0.9445750230732326E+00$ $\pm 0.9894009349916499E+00$	0.1894506104550685E+00 0.1826034150449236E+00 0.1691565193950024E+00 0.1495959888165733E+00 0.1246289712555339E+00 0.9515851168249290E-01 0.6225352393864778E-01 0.2715245941175185E-01

GQ summary

- Requires function evaluations at non-uniformly spaced points within the integration interval
 - not appropriate for cases where the function is unknown
 - not suited for dealing with tabulated data that appear in many engineering problems
- If the function is known, its efficiency can be a decided advantage

Summary Example

Consider
$$I = \int_{-1}^1 \frac{dx}{x+2} = \ln(3) - \ln(1) \approx 1.09861$$

Approximations

- rectangular rule:
$$\mathcal{I}_R = 1 \cdot \frac{1}{2} = 0.5$$
- trapezoidal rule:
$$\mathcal{I}_T = \frac{2}{2} \left(\frac{1}{1} + \frac{1}{3} \right) = \frac{4}{3} = 1.333 \dots$$
- Simpson rule:
$$\mathcal{I}_S = \frac{1}{3} \left(\frac{1}{1} + \frac{4}{2} + \frac{1}{3} \right) = \frac{10}{9} = 1.111 \dots$$
- Gauss-Legendre 2nd order:
$$\mathcal{I}_{GL} = \frac{1}{-\frac{1}{\sqrt{3}} + 2} + \frac{1}{\frac{1}{\sqrt{3}} + 2} = 1.090909 \dots$$

Demo: adaptive Simpson

Numerical recipes: “with the exception of two of the most modest formulas (“extended trapezoidal rule,” and “extended midpoint rule,”), the classical formulas are almost entirely useless. They are museum pieces, but beautiful ones.”