Computational Physics

Kepler Problem

- Model
- Numerical Treatment

Model

- The Kepler problem is a special case of the two-body problem.
- two-body potential $U(|\mathbf{r_1}\text{-}\mathbf{r_2}|)$ → Lagrange function:

$$L(r_1, r_2, p_1, p_2) = \frac{p_1^2}{2m_1} + \frac{p_2^2}{2m_2} - U(|r_1 - r_2|)$$

- reducible to two-dimensional motion of point particle with reduce mass m in central potential U
- Energy, E, and angular momentum, ℓ, are conserved
- rotational symmetry → use polar coordinates (ρ,φ)

Equations of motion (EoM) for relative coordinate:

(see Appendix A for derivation)

$$\dot{\varphi} = \frac{\ell}{m\rho^2}$$

$$\dot{\rho} = \pm\sqrt{\frac{2}{m}\left[E - U(\rho) - \frac{\ell^2}{2m\rho^2}\right]}$$

...

- define effective potential: $$U_{\mathrm{eff}}(\rho) = U(\rho) + \frac{\ell^2}{2m\rho^2}$$

ℓ: angular momentum

U_{mom} "centrifugal barrier"

- Separation of variables in the EoM for ρ yields:

$$t = t_0 \pm \int_{\rho_0}^{\rho} \mathrm{d}\rho' \left\{ \frac{2}{m} \left[E - U_{\mathrm{eff}}(\rho') \right] \right\}^{-\frac{1}{2}}$$

where $\rho_0 \equiv \rho(t_0)$ is the initial condition at time t_0

- similarly for φ with $dt = d\rho/(d\rho/dt)$:

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\rho} = \frac{\mathrm{d}\varphi}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\rho} = \pm\frac{|\ell|}{m\rho^2}\left[\frac{2}{m}\left(E - U(\rho) - \frac{|\ell|^2}{2m\rho^2}\right)\right]^{-\frac{1}{2}}$$

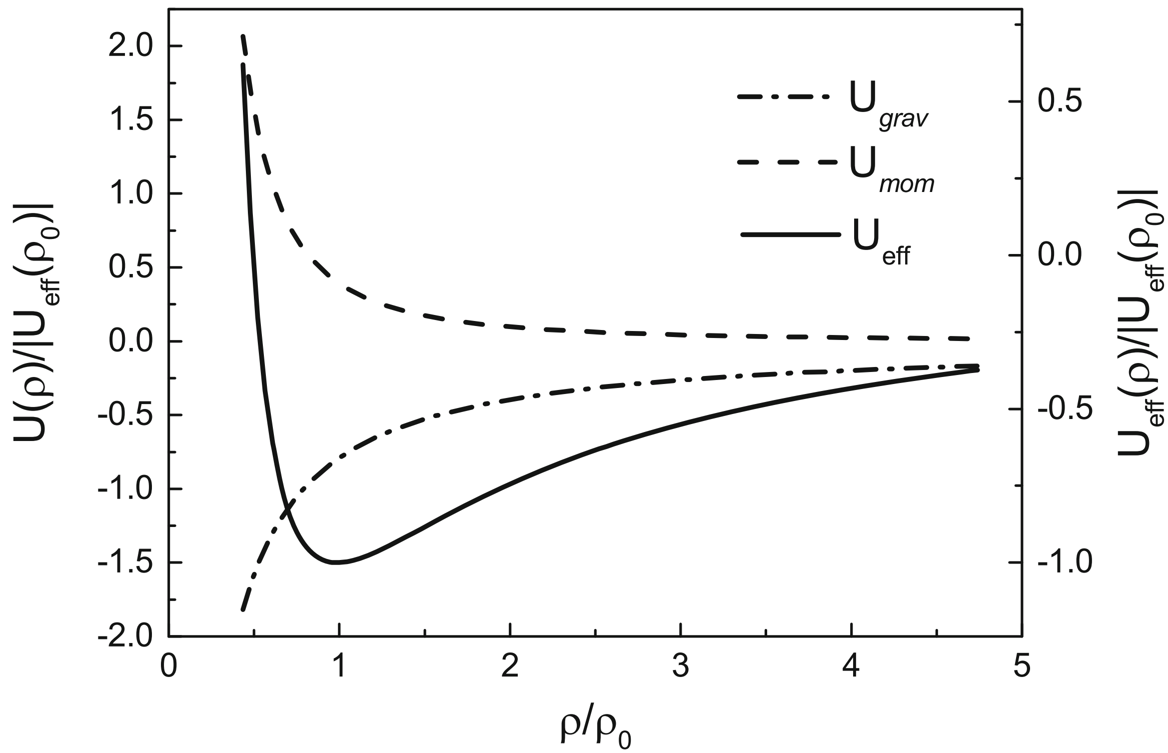
$$\rightarrow \quad \varphi = \varphi_0 \pm \int_{\rho_0}^{\rho} \mathrm{d}\rho' \frac{\ell}{m\rho'^2} \left\{ \frac{2}{m} \left[E - U_{\mathrm{eff}}(\rho') \right] \right\}^{-\frac{1}{2}}$$

with $\varphi_0 = \varphi(t_0)$

potential

For the Kepler problem we use the gravitational potential:

$$U(\rho) = -\frac{\alpha}{\rho}, \quad \alpha > 0$$



solution for φ

Using the gravitational potential, we have to calculate:

$$\varphi = \varphi_0 \pm \int_{\rho_0}^{\rho} \mathrm{d}\rho' \frac{\ell}{m\rho'^2} \left[\frac{2}{m}\left(E + \frac{\alpha}{\rho'} - \frac{\ell^2}{2m\rho'^2}\right)\right]^{-\frac{1}{2}}$$

with substitution u=1/ρ

$$\varphi = \varphi_0 \mp \int_{u_1}^{u_2} \mathrm{d}u \left[\frac{2mE}{\ell^2} + \frac{2m\alpha}{\ell^2}u - u^2\right]^{-\frac{1}{2}}$$

This integral can be calculated and results in

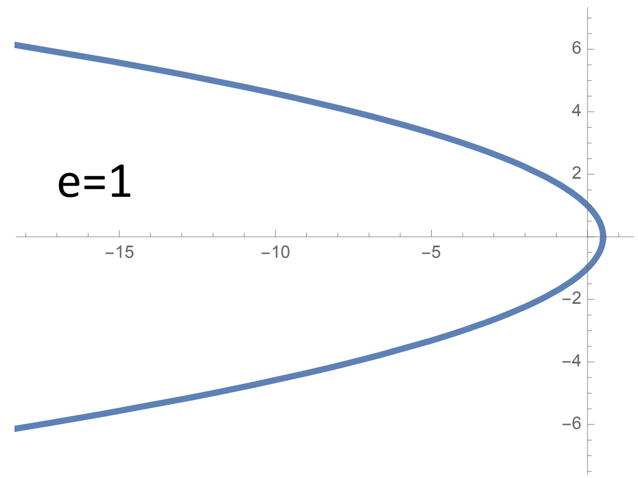
$$\varphi = \varphi_0 \pm \cos^{-1}\left(\frac{\frac{\ell}{\rho} - \frac{m\alpha}{\ell}}{\sqrt{2mE + \frac{m^2\alpha^2}{\ell^2}}}\right) + \mathrm{const}$$

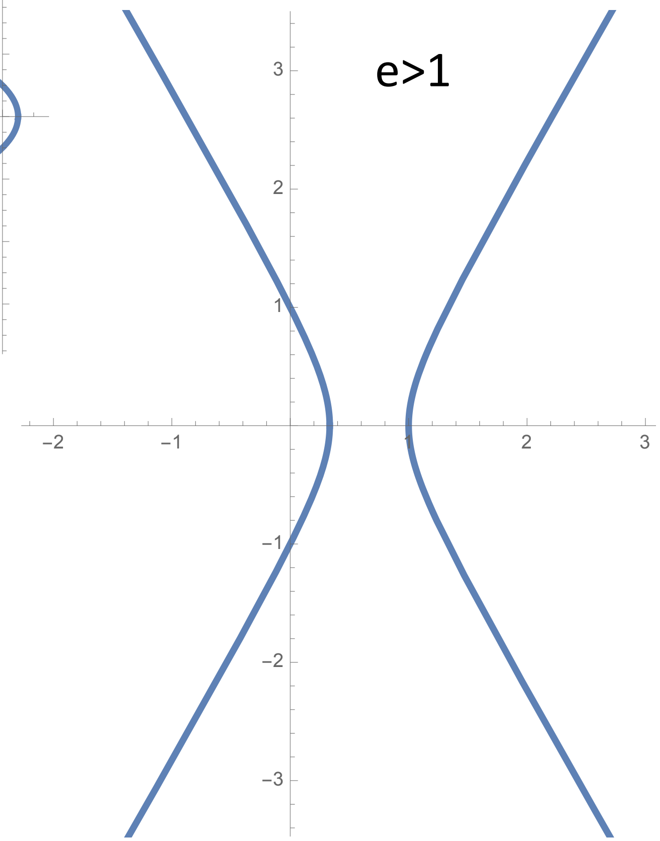
With $a = \frac{\ell^2}{m\alpha}$ and eccentricity $e = \sqrt{1 + \frac{2E\ell^2}{m\alpha^2}}$ and const=0, φ_0=0, we get

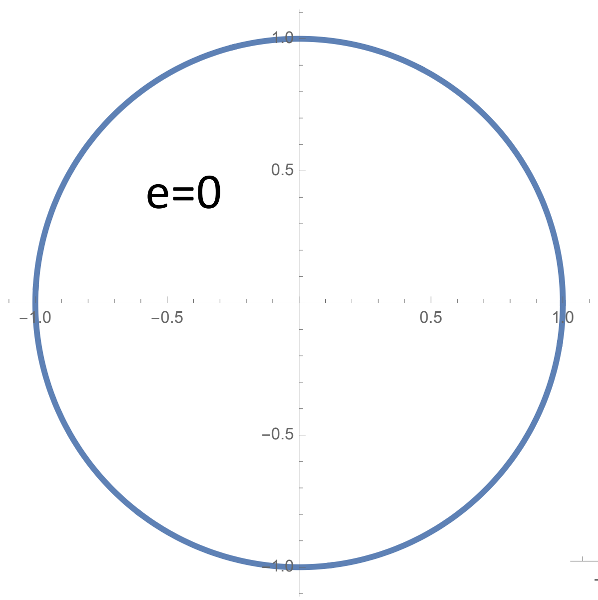
$$\boxed{\frac{a}{\rho} = 1 + e\cos(\varphi)}$$

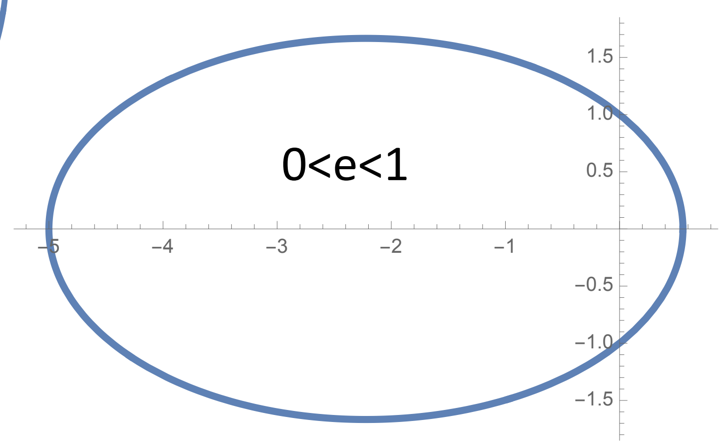
Different type of solutions

- we can write the solution for ρ as $\rho(\varphi)=a/(1+e\cos(\varphi))$
- and in rectangular coordinates
  $x(\rho,\varphi)=\rho(\varphi)\cos(\varphi)$
  $y(\rho,\varphi)=\rho(\varphi)\sin(\varphi)$









Numerical Treatment

Before, we solved the Kepler problem analytically and expressed φ as function of ρ or inversely ρ as function of φ

Here we solve

$$t = t_0 \pm \int_{\rho_0}^{\rho} \mathrm{d}\rho' \left\{ \frac{2}{m} \left[E - U_{\mathrm{eff}}(\rho') \right] \right\}^{-\frac{1}{2}}$$

numerically. We rewrite:

$$t - t_0 = \int_{\rho_0}^{\rho} \mathrm{d}\rho' f(\rho')$$

and discretize time in $t_n = t_0 + n\Delta t$ with $\rho_n \equiv \rho(t_n)$. So for a timestep unit we can write

$$\Delta t = t_n - t_{n-1} = \int_{\rho_n}^{\rho_{n+1}} \mathrm{d}\rho' f(\rho')$$

which is using the rectangular rule:

$$\boxed{\Delta t = (\rho_{n+1} - \rho_n) f(\rho_n)}$$

...

Solving for ρ_{n+1} gives $\boxed{\rho_{n+1} = h(\rho_n)\Delta t + \rho_n}$ $\longleftrightarrow$ $D_+\rho_n = h(\rho_n)$

with
$$h(\rho) = \frac{1}{f(\rho)} = \sqrt{\frac{2}{m}\left[E - U_{\mathrm{eff}}(\rho)\right]}$$

Since the rhs of above eq. does not depend on ρ_{n+1} it is called an explicit method.

An ODE of form $\dot{y} = F(y)$ approximated as $y_{n+1} = y_n + F(y_n)\Delta t$
is called explicit Euler scheme (as we used for the pendulum).
Using the backward rectangular rule for the ρ integral gives:

$$t_{n+1} - t_n = (\rho_{n+1} - \rho_n) f(\rho_{n+1})$$

or
$$\boxed{\rho_{n+1} = \rho_n + h(\rho_{n+1})\Delta t} \longleftrightarrow D_-\rho_{n+1} = h(\rho_{n+1})$$

which is an implicit method. $y_{n+1} = y_n + F(y_{n+1})\Delta t$ Implicit Euler method

Implicit method

In general, the implicit Euler method is not analytically solvable → need numerical methods (implies numerical error)

For the the Kepler problem we can solve it analytically since it is a 4th order polynomial:

$$\rho_{n+1}^4 - 2\rho_n\rho_{n+1}^3 + \left(\rho_n^2 - \frac{2E\Delta t^2}{m}\right)\rho_{n+1}^2 - \frac{2\alpha\Delta t^2}{m}\rho_{n+1} + \frac{\ell^2\Delta t^2}{m^2} = 0$$

but it is tedious (see Ferrari's method).

We can also use the central rectangular rule:

$$\Delta t = (\rho_{n+1} - \rho_n) f\left(\frac{\rho_{n+1} + \rho_n}{2}\right)$$

or

$$\boxed{\rho_{n+1} = \rho_n + h\left(\frac{\rho_{n+1} + \rho_n}{2}\right)\Delta t} \longleftrightarrow D_c\rho_{n+\frac{1}{2}} = h\left(\frac{\rho_{n+1} + \rho_n}{2}\right)$$

in general:

$$y_{n+1} = y_n + F\left(\frac{y_{n+1} + y_n}{2}\right)\Delta t$$

implicit midpoint rule

errors

The error of the forward (explicit) and backward (implicit) rectangular rules is of order Δt, while being Δt^2 for the implicit midpoint rule.
In any case the consequence is that energy and angular momentum will not be conserved. I.e. deviations from the analytical trajectories can be expected!

→ Lab Thursday

Implement explicit method to solve for $\mathbf{r}(\varphi)$ and $\mathbf{r}(t)$

Use:
- python
- C/C++

Plotting tools (suggestion):
- gnuplot, LabPlot
- Jupyter

Read Appendix A for more details of the two-body problem.