

Computational Physics

Double Pendulum

- Hamilton's Equation
- Numerical Solution
- Numerical Analysis of Chaos

Equation of motion

1. generalized coordinates
2. Lagrange function for 2D double pendulum
3. generalized momenta & Hamilton function

coordinates for the two point masses:

$$x_1 = \ell \sin(\varphi_1)$$

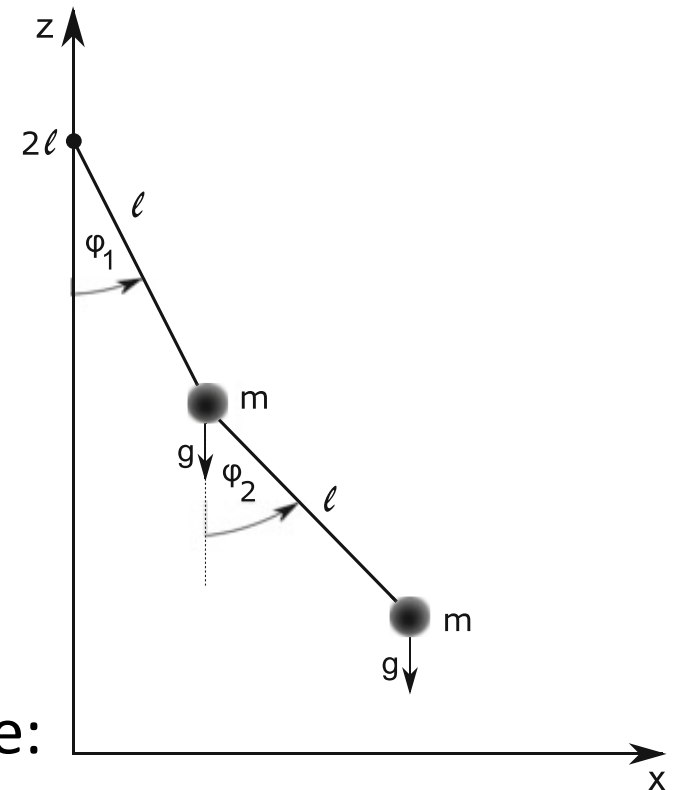
$$z_1 = 2\ell - \ell \cos(\varphi_1)$$

$$x_2 = \ell [\sin(\varphi_1) + \sin(\varphi_2)]$$

$$z_2 = 2\ell - \ell [\cos(\varphi_1) + \cos(\varphi_2)]$$

Here both length (and masses) are the same:

→ total length $2\ell = \text{const.}$ (for simulation introduce m_1, m_2 and ℓ_1, ℓ_2)



Time derivatives and energies

- using the above coordinates, we get:

$$\dot{x}_1 = \ell \dot{\varphi}_1 \cos(\varphi_1) ,$$

$$\dot{z}_1 = \ell \dot{\varphi}_1 \sin(\varphi_1) ,$$

$$\dot{x}_2 = \ell [\dot{\varphi}_1 \cos(\varphi_1) + \dot{\varphi}_2 \cos(\varphi_2)]$$

$$\dot{z}_2 = \ell [\dot{\varphi}_1 \sin(\varphi_1) + \dot{\varphi}_2 \sin(\varphi_2)]$$

- Reminder Lagrange function (Lagrangian): $L=T-U$, here kinetic energy

$$T = \frac{m}{2} (\dot{x}_1^2 + \dot{z}_1^2 + \dot{x}_2^2 + \dot{z}_2^2)$$

$$\sin(x) \sin(y) + \cos(x) \cos(y) = \cos(x - y)$$

$$= \frac{m\ell^2}{2} [2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)]$$

potential energy $U = mgz_1 + mgz_2$

$$= mg\ell [4 - 2 \cos(\varphi_1) - \cos(\varphi_2)]$$

Lagrangian

Therefore, the Lagrangian for the double pendulum is

$$L = \frac{m\ell^2}{2} [2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)] - mg\ell [4 - 2\cos(\varphi_1) - \cos(\varphi_2)]$$

From this we get the generalized momenta:

$$p_1 = \frac{\partial}{\partial \dot{\varphi}_1} L = m\ell^2 [2\dot{\varphi}_1 + \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)]$$

$$p_2 = \frac{\partial}{\partial \dot{\varphi}_2} L = m\ell^2 [\dot{\varphi}_2 + \dot{\varphi}_1 \cos(\varphi_1 - \varphi_2)]$$

To get the Hamilton function (Hamiltonian), we need to express the kinetic energy in term of the generalized momenta p_1 & p_2

kinetic energy rewrite

Using the expression for p_1 and p_2 , we express $d\varphi_1/dt$ and $d\varphi_2/dt$ in terms of p_1 and p_2

- first we solve the expression for p_2 for $\dot{\varphi}_2$:

$$\dot{\varphi}_2 = \frac{p_2}{m\ell^2} - \dot{\varphi}_1 \cos(\varphi_1 - \varphi_2)$$

- which we then insert in the p_1 expression:

$$\dot{\varphi}_1 = [2 - \cos^2(\varphi_1 - \varphi_2)]^{-1} \left[\frac{p_1}{m\ell^2} - \frac{p_2}{m\ell^2} \cos(\varphi_1 - \varphi_2) \right]$$

which rewrites to

$$\dot{\varphi}_1 = \frac{1}{m\ell^2} \frac{p_1 - p_2 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)}$$

and similarity

$$\begin{aligned} \dot{\varphi}_2 &= \frac{1}{m\ell^2} \left[p_2 - \frac{p_1 \cos(\varphi_1 - \varphi_2) - p_2 \cos^2(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)} \right] \\ &= \frac{1}{m\ell^2} \frac{2p_2 - p_1 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)}. \end{aligned}$$

...

- this gives

$$\begin{aligned} T &= \frac{m\ell^2}{2} [2\dot{\varphi}_1^2 + \dot{\varphi}_2^2 + 2\dot{\varphi}_1\dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)] \\ &= \frac{1}{2m\ell^2} \frac{p_1^2 + 2p_2^2 - 2p_1p_2 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)} . \end{aligned}$$

The Hamiltonian then follows from $H=T+U$:

$$\begin{aligned} H &= T + U \\ &= \frac{1}{2m\ell^2} \frac{p_1^2 + 2p_2^2 - 2p_1p_2 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)} \\ &\quad + mgl [4 - 2 \cos(\varphi_1) - \cos(\varphi_2)] . \end{aligned}$$

Equations of motion

Now we can write the Hamilton equations of motion:

$$\dot{\varphi}_i = \frac{\partial}{\partial p_i} H, \quad \dot{p}_i = -\frac{\partial}{\partial \varphi_i} H, \quad i = 1, 2$$

which results in the following coupled differential equations:

$$\begin{aligned} \dot{\varphi}_1 &= \frac{1}{m\ell^2} \frac{p_1 - p_2 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)} & \dot{p}_1 &= \frac{1}{m\ell^2} \frac{1}{1 + \sin^2(\varphi_1 - \varphi_2)} \left[-p_1 p_2 \sin(\varphi_1 - \varphi_2) \right. \\ & & & \left. + \frac{p_1^2 + 2p_2^2 - 2p_1 p_2 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)} \cos(\varphi_1 - \varphi_2) \sin(\varphi_1 - \varphi_2) \right] \\ \dot{\varphi}_2 &= \frac{1}{m\ell^2} \frac{2p_2 - p_1 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)} & & -2mg\ell \sin(\varphi_1), \\ & & \dot{p}_2 &= \frac{1}{m\ell^2} \frac{1}{1 + \sin^2(\varphi_1 - \varphi_2)} \left[p_1 p_2 \sin(\varphi_1 - \varphi_2) \right. \\ & & & \left. - \frac{p_1^2 + 2p_2^2 - 2p_1 p_2 \cos(\varphi_1 - \varphi_2)}{1 + \sin^2(\varphi_1 - \varphi_2)} \sin(\varphi_1 - \varphi_2) \cos(\varphi_1 - \varphi_2) \right] \\ & & & -mg\ell \sin(\varphi_2). \end{aligned}$$

Numerical Solution

We will use RK4 to solve these equations numerically.

We rewrite using the 4D vector:

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \equiv \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ p_1 \\ p_2 \end{pmatrix}$$

With this the EoM have the form:
$$\begin{pmatrix} \dot{\varphi}_1 \\ \dot{\varphi}_2 \\ \dot{p}_1 \\ \dot{p}_2 \end{pmatrix} = F(y) \equiv \begin{pmatrix} f_1(y) \\ f_2(y) \\ f_3(y) \\ f_4(y) \end{pmatrix}$$

and we discretize the time as usual $t_n = n\Delta t$, $y_n = y(t_n)$

$$Y_1 = y_n ,$$

$$Y_2 = y_n + \frac{\Delta t}{2} F(Y_1) ,$$

$$Y_3 = y_n + \frac{\Delta t}{2} F(Y_2) ,$$

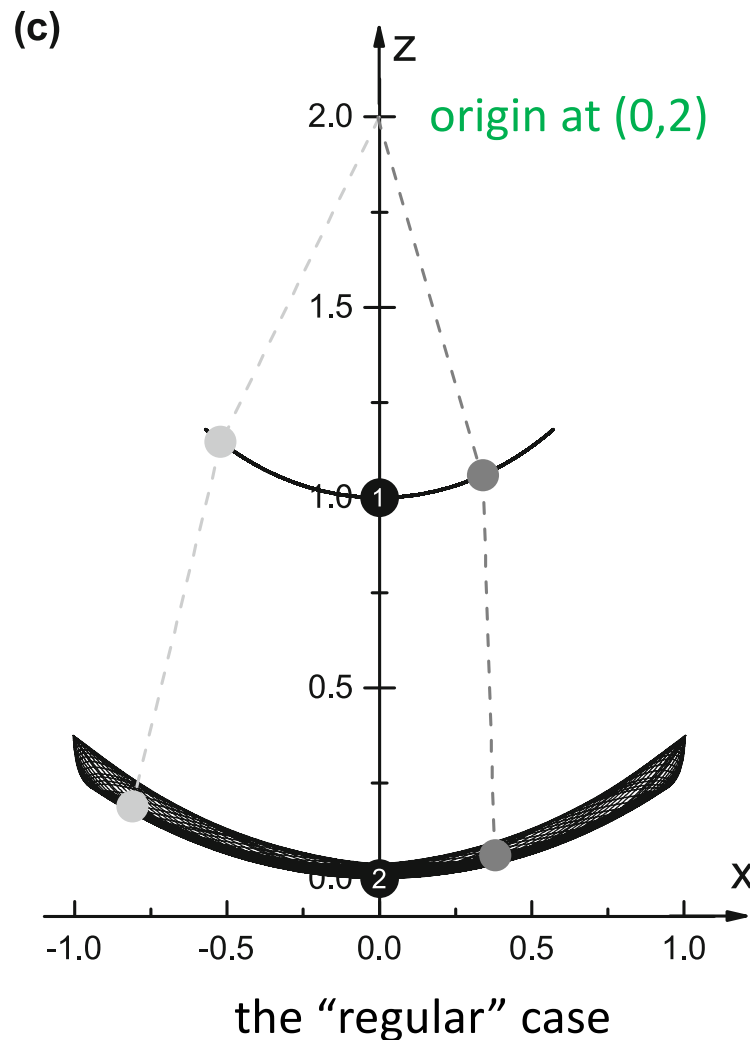
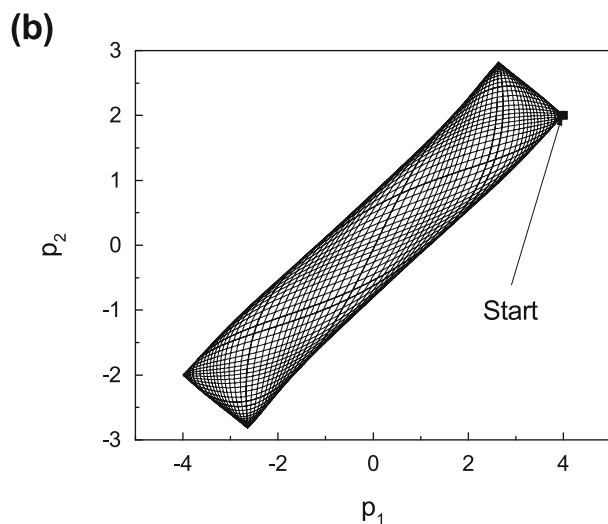
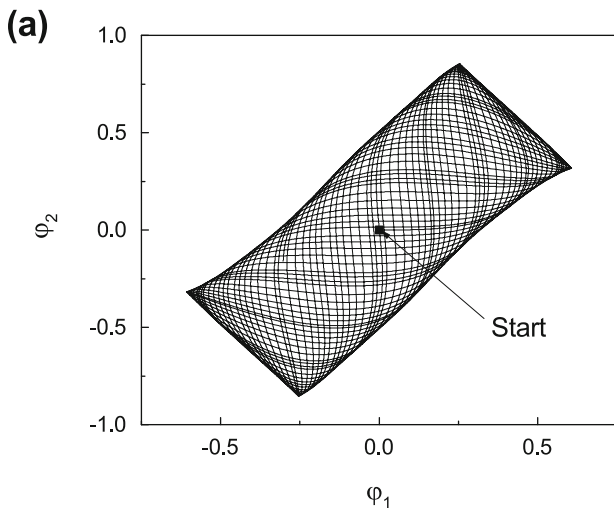
$$Y_4 = y_n + \Delta t F(Y_3) ,$$

$$y_{n+1} = y_n + \frac{\Delta t}{6} [F(Y_1) + 2F(Y_2) + 2F(Y_3) + F(Y_4)]$$

RK4:

Solutions (I)

Using the RK4 scheme we can expect the following solutions



initial conditions:

$$\varphi_1(0) = \varphi_2(0) = 0$$

$$p_1(0) = 4.0$$

$$p_2(0) = 2.0$$

Parameters

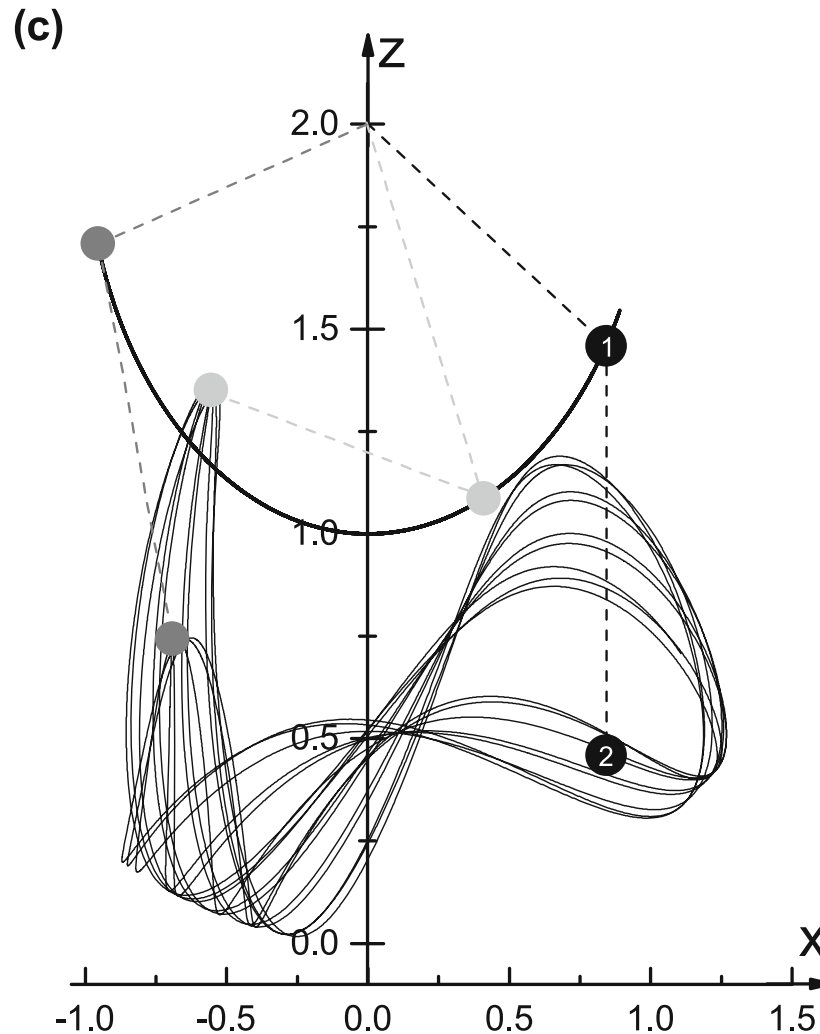
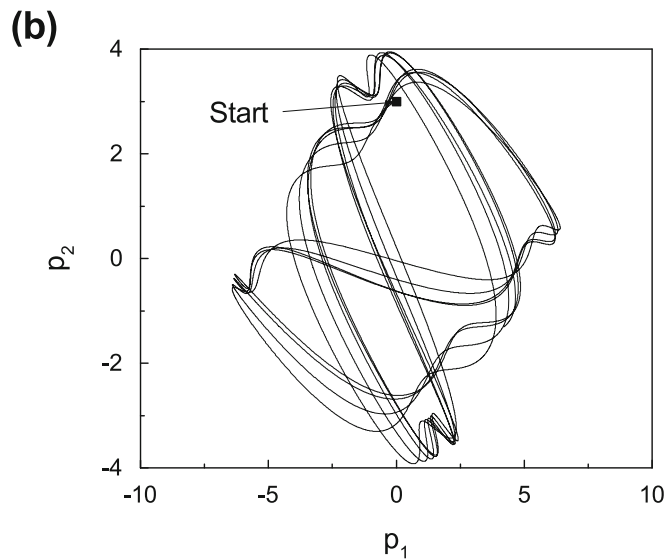
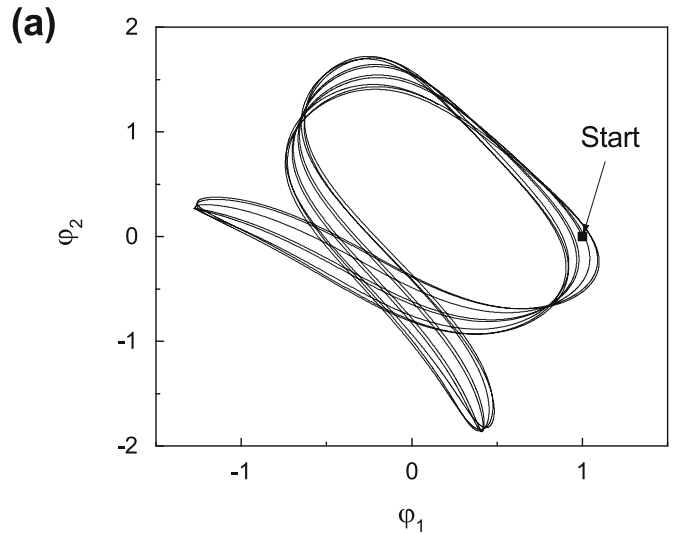
$$m = 1\text{kg}, \ell = 1\text{m}$$

$$g = 9.8067\text{m/s}^2$$

$$\Delta t = 0.001$$

$$N = 60,000$$

... (II)



initial conditions:

$$\phi_1(0)=1.0$$

$$\phi_2(0)=0$$

$$p_1(0)=0.0$$

$$p_2(0)=3.0$$

Parameters

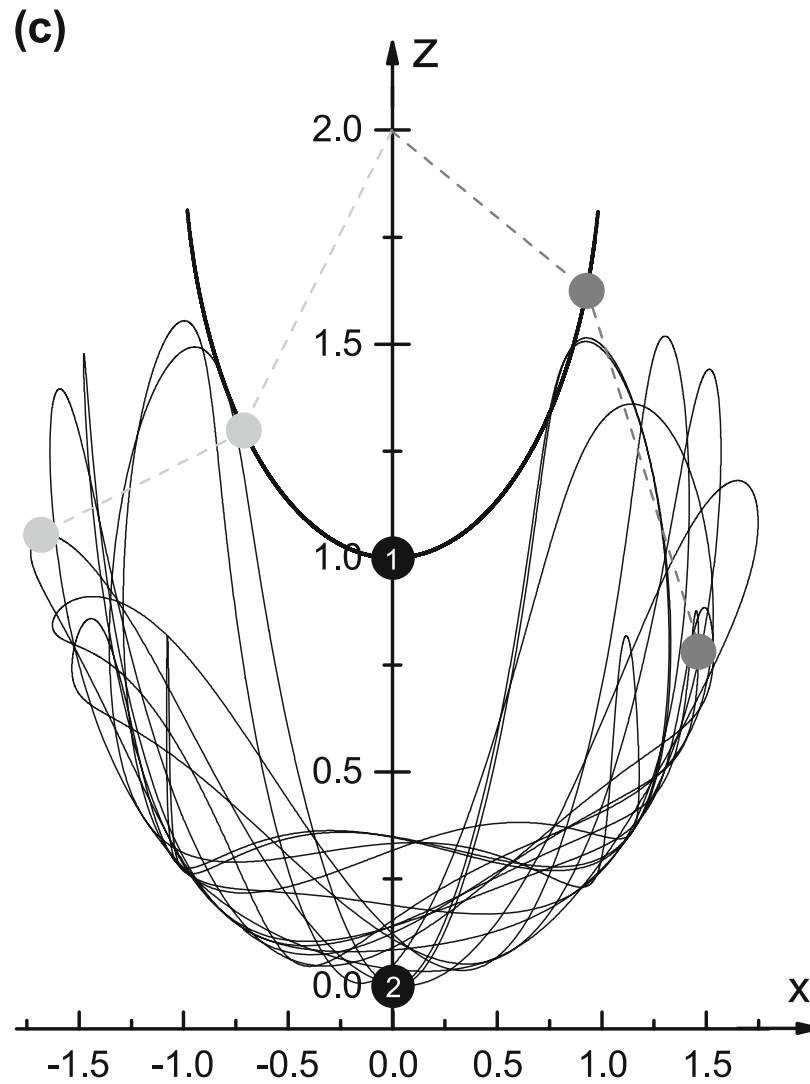
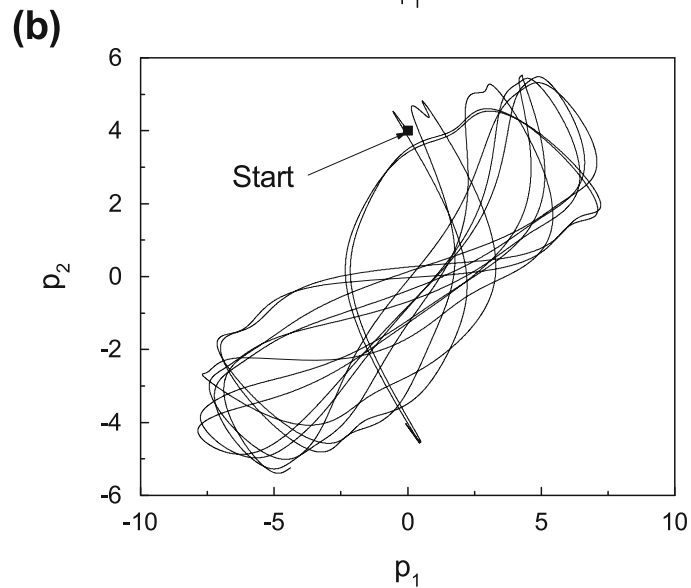
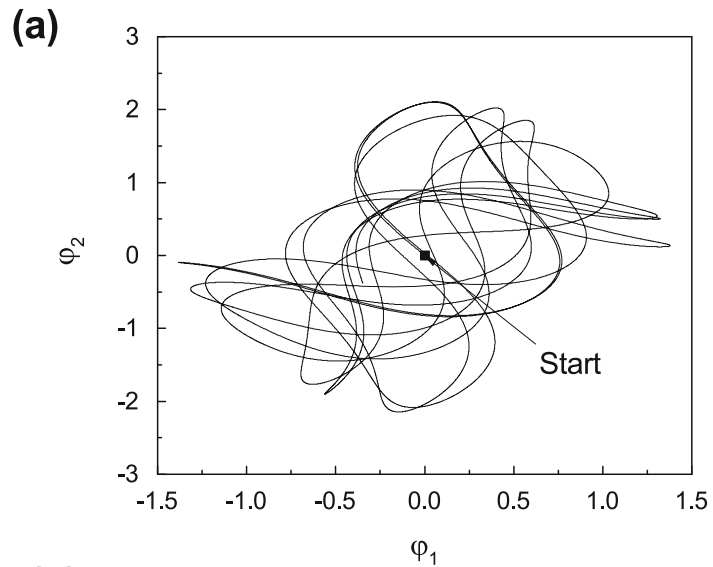
$$m=1\text{kg}, \ell=1\text{m}$$

$$g=9.8067\text{m/s}^2$$

$$\Delta t=0.001$$

$$N=60,000$$

... (III)



initial conditions:

$$\phi_1(0)=\phi_2(0)=0$$

$$p_1(0)=0.0$$

$$p_2(0)=4.0$$

Parameters

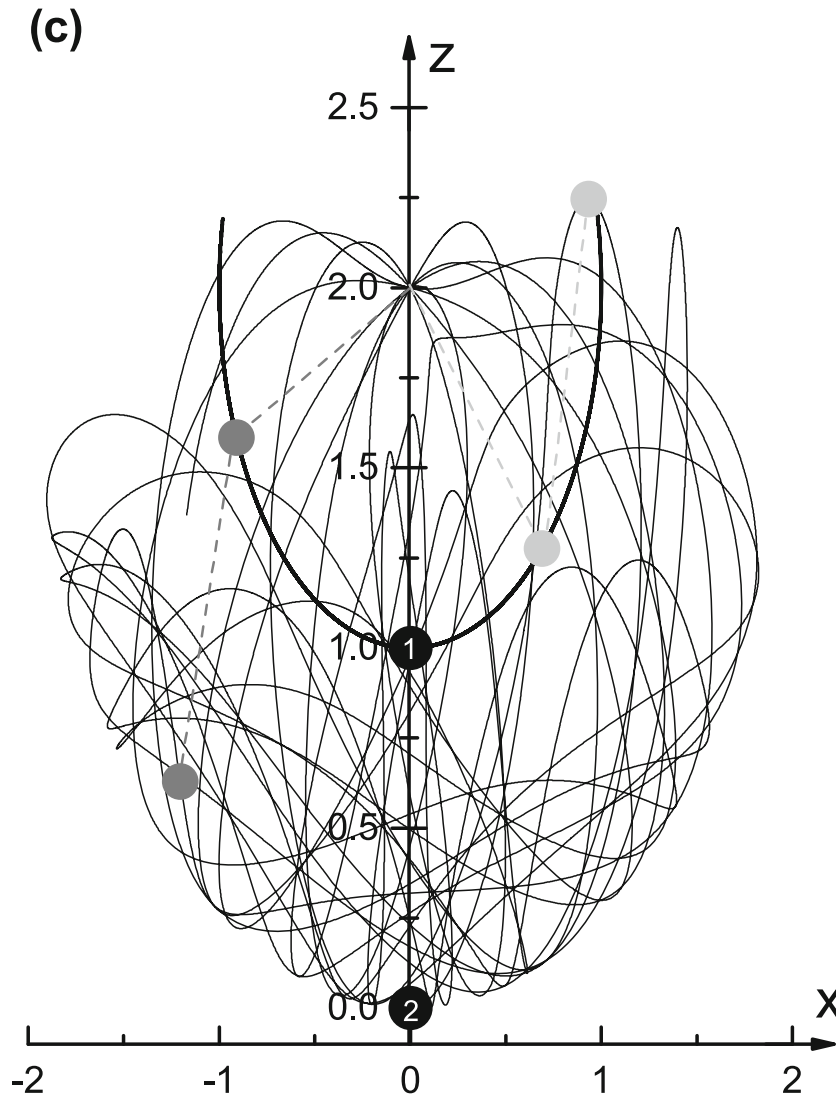
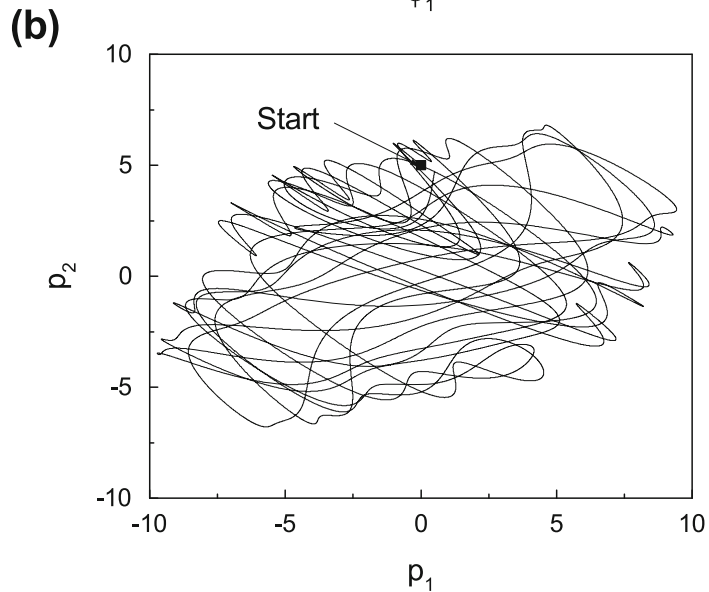
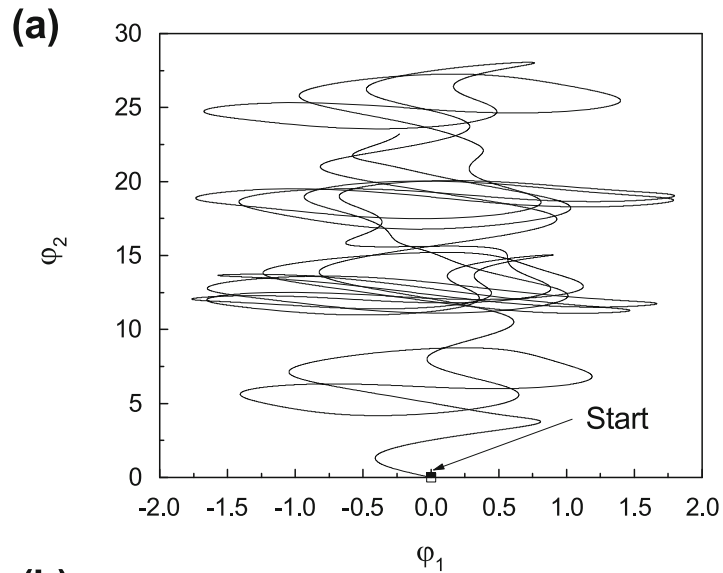
$$m=1\text{kg}, \ell=1\text{m}$$

$$g=9.8067\text{m/s}^2$$

$$\Delta t=0.001$$

$$N=60,000$$

... (IV)



initial conditions:

$$\varphi_1(0)=\varphi_2(0)=0$$

$$p_1(0)=0.0$$

$$p_2(0)=5.0$$

Parameters

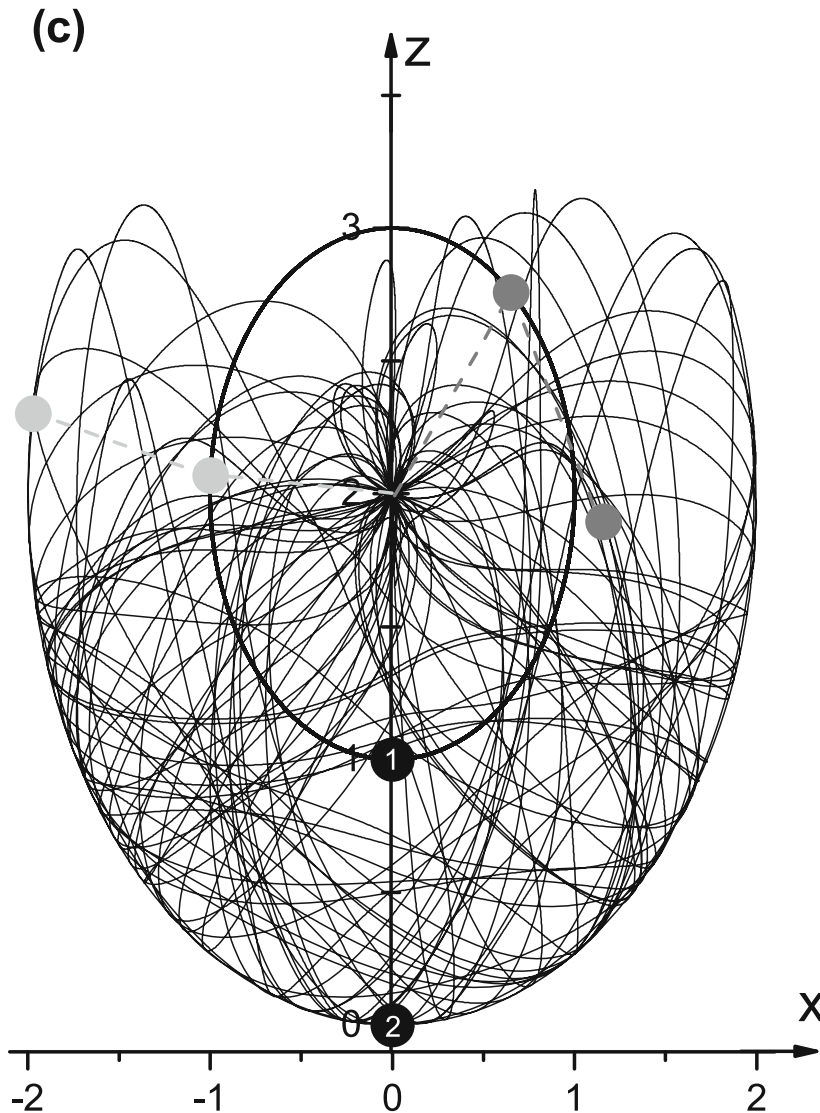
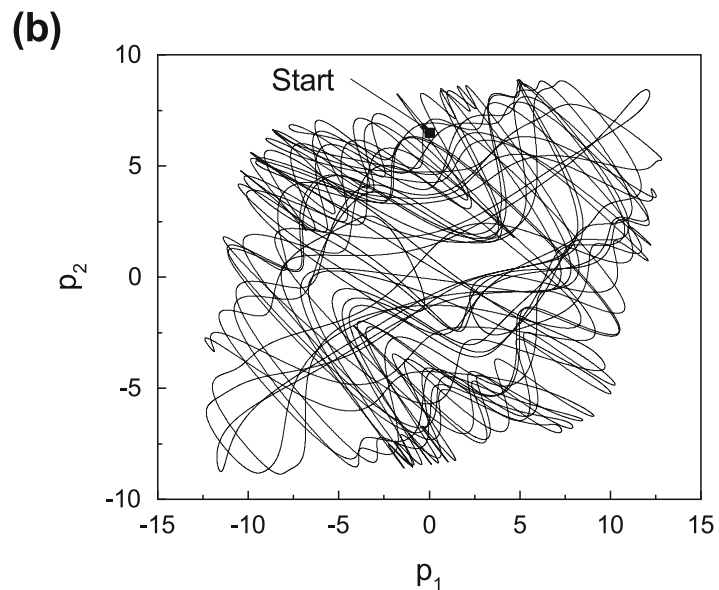
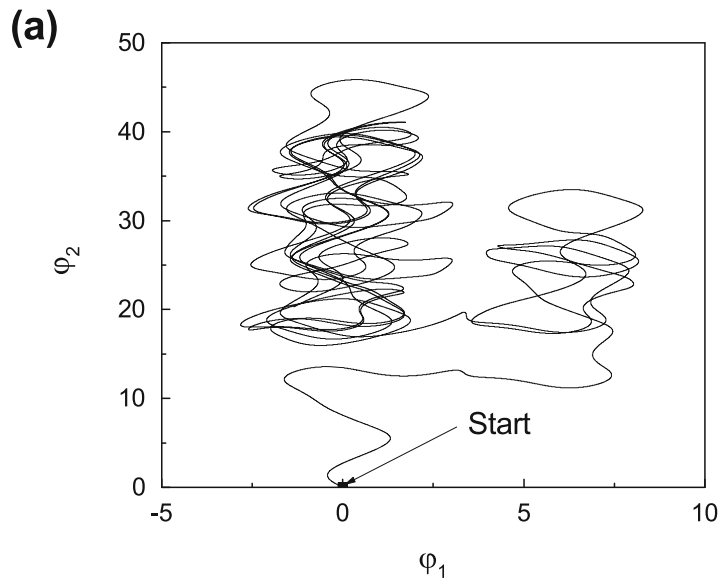
$$m=1\text{kg}, \ell=1\text{m}$$

$$g=9.8067\text{m/s}^2$$

$$\Delta t=0.001$$

$$N=60,000$$

... (V)



initial conditions:

$$\varphi_1(0)=\varphi_2(0)=0$$

$$p_1(0)=0.0$$

$$p_2(0)=6.5$$

Parameters

$$m=1\text{kg}, \ell=1\text{m}$$

$$g=9.8067\text{m/s}^2$$

$$\Delta t=0.001$$

$$N=60,000$$

"chaotic" behavior

Numerical Analysis of Chaos

Notation:

system with f degrees of freedom at time t completely determined by:

- generalized coordinates: $q_1(t), \dots, q_f(t)$
- generalized momenta: $p_1(t), \dots, p_f(t)$

The vector $\mathbf{q}(t) = (q_1(t), \dots, q_f(t))^T$ is a point in *configuration space*.

For the double pendulum: $f=2$ and $q_i(t) = \varphi_i(t) \in (-\pi, \pi]$, $i=1,2$

The $2f$ -dimensional vector $\mathbf{x}(t) = (q_1(t), \dots, q_f(t), p_1(t), \dots, p_f(t))^T$ is a point in phase-space and the time evolution of the system is described by the phase-space trajectory $\mathbf{x}(t)$.

In an *autonomous* (or time-invariant) system the total energy is conserved

$$\frac{\partial}{\partial t} H(x, t) = 0$$

...

A system is *integrable* if f independent invariants $I_1, \dots, I_f$ exist (one would be the energy)

$$I_j(x) = I_j = \text{const}, \quad j = 1, \dots, f$$

A 1D autonomous system is always integrable ($I_1=E$)

Non-integrable systems can show chaotic behavior.

- trajectories can be strongly dependence on the initial conditions
- makes an analytic calculation of the dynamics extremely difficult.

However, since the trajectories can be computed without problems by numeric means, we discuss now how to characterize chaotic behavior on the computer.

Analysis of autonomous systems

- We analyze the system by looking at two different initial conditions $\mathbf{x}_0$ and $\mathbf{x}'_0$
- After time t , these evolved into $\mathbf{x}(t)$ and $\mathbf{x}'(t)$
- We define the distance in phase-space $d(t) = |\mathbf{x}(t) - \mathbf{x}'(t)|$ with some norm $|\cdot|$.

A solution $\mathbf{x}(t)$ is *stable* if

$$\forall \epsilon > 0 \exists \delta(\epsilon) > 0 : \forall \mathbf{x}'_0 : d(0) < \delta \Rightarrow d(t) < \epsilon, \quad \forall t > 0$$

or

A solution is stable if the trajectory $\mathbf{x}'(t)$, corresponding to the perturbed initial condition $\mathbf{x}'_0$, stays within a tube of radius ϵ around the unperturbed trajectory $\mathbf{x}(t)$ for all $t > 0$

- Asymptotically stable: $d(t) \rightarrow 0$ for $t \rightarrow \infty$
These solutions attract neighboring solutions: They are *attractors*
- A solution is a *periodic orbit* if $\mathbf{x}(t) = \mathbf{x}(t + \tau)$ for all $\mathbf{x}(t)$

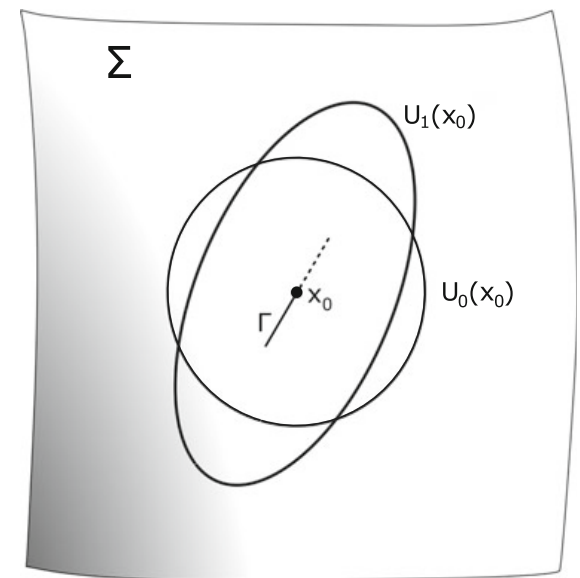
Poincaré maps

To study non-integrable system, it is useful to use a topological method: Poincaré maps:

- Study only intersection point of $\mathbf{x}(t)$ with a $2f-1$ dimensional manifold Σ in the $2f$ -dimensional phase-space
- Σ is transverse to $\mathbf{x}(t)$, i.e., $\mathbf{x}(t)$ is not parallel or within Σ
- For periodic or, more general, bound trajectories, $\mathbf{x}(t)$ has multiple intersections with Σ
- A **Poincaré map** is then the mapping of one intersection point to the next

For example:

- if $\mathbf{x}_0 \in \Sigma$ and $\mathbf{x}(t)$ is periodic, the next intersection point $\mathbf{x}_1 = \mathbf{x}(\tau) = \mathbf{x}_0$.
- for a slightly perturbed $\mathbf{x}'_0 \in U_0(\mathbf{x}_0)$, the trajectory will in general not be quite periodic and the next intersection point $\mathbf{x}'_1 \neq \mathbf{x}'_0$. $\mathbf{x}'_1 \in U_1(\mathbf{x}_0)$
- The map $P(\mathbf{x}'_0) = \mathbf{x}'_1$ is the Poincaré map.
[above $P(\mathbf{x}_0) = \mathbf{x}_0$: a fixed point]
- $U_0(\mathbf{x}_0)$ is a “neighborhood” region of $\mathbf{x}_0$, $U_1(\mathbf{x}_0)$ the neighborhood of the first return



Poincaré plots

Two possible scenarios for **integrable** systems

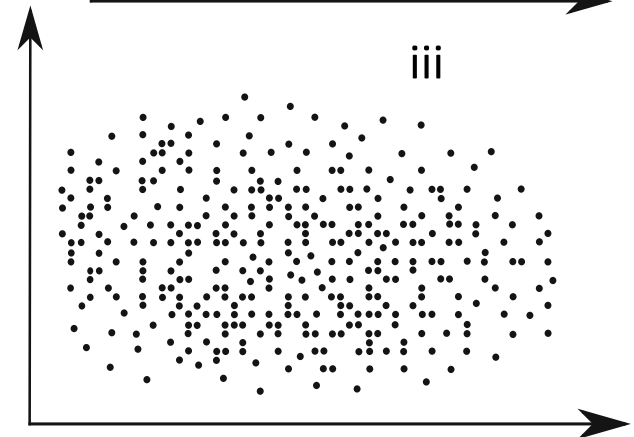
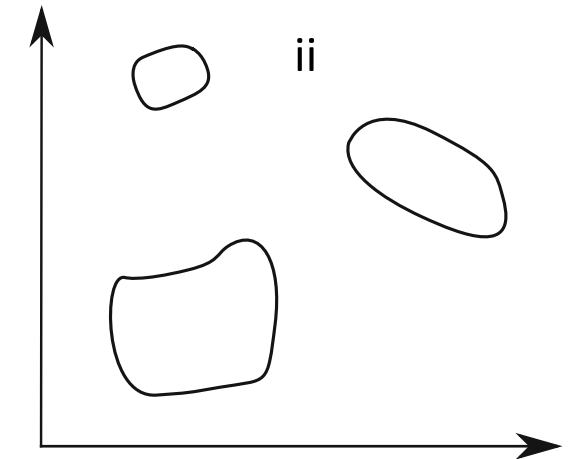
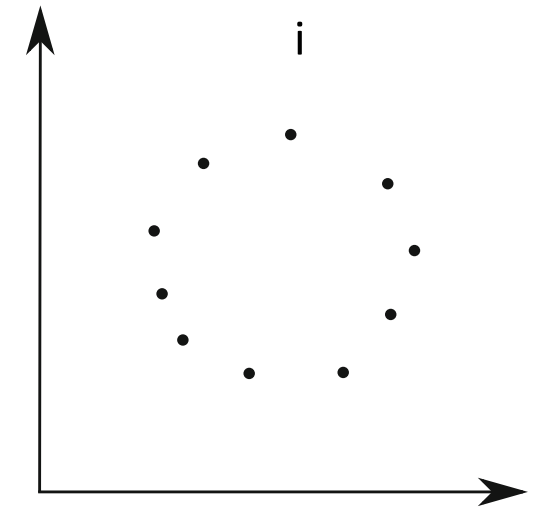
- i. number of intersection points is finite
- ii. or infinite

For **non-integrable** systems a third case is possible:

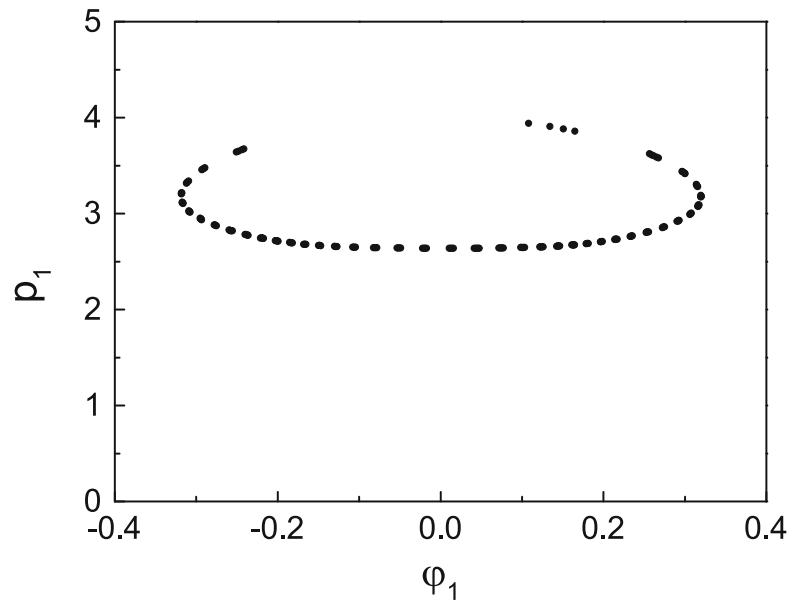
iii. chaotic behavior: intersection points are space filling

Case of the double pendulum:

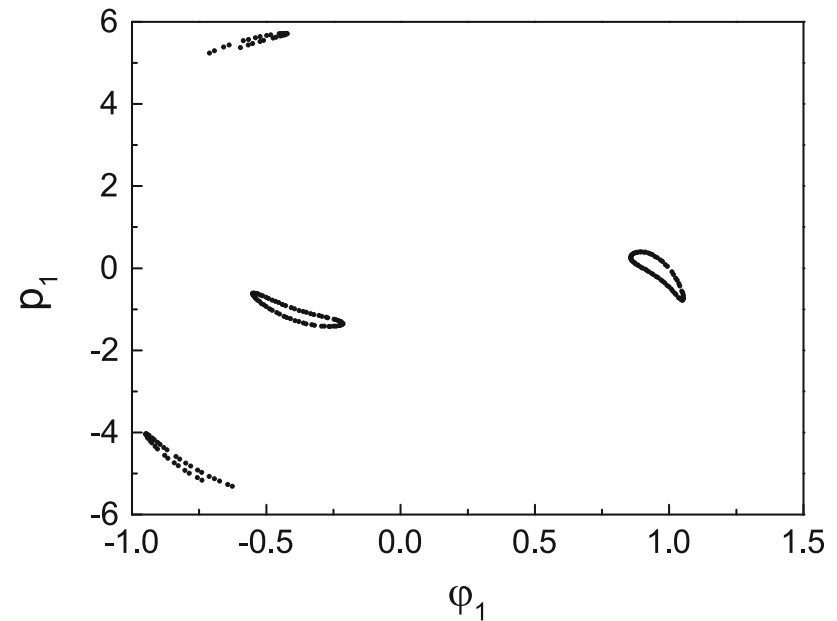
- Since $E = \text{const}$, the available phase-space is a three-dimensional manifold in the $(\varphi_1, \varphi_2, p_1, p_2)$ -space
- define the 2D Poincare section Σ as $\varphi_2(t) = 0$ and $p_2 > 0$
- intersection points form lines in Σ , either disconnected (case i) or connected (case ii)



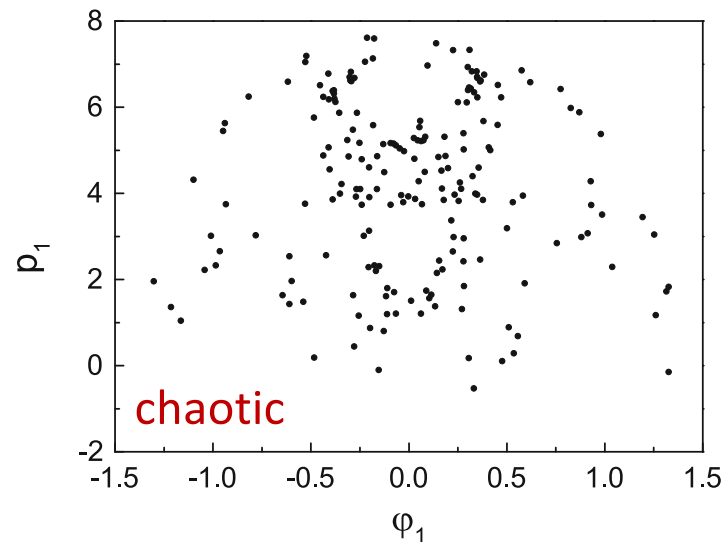
Double pendulum: Poincaré plots



corresponds to “regular” case (I)



(II)



(III)

Lab (today &) Thursday

Implement RK4 for the double pendulum

Reproduce the different solutions in φ_1 and φ_2 space, p_1 and p_2 space and real space trajectories

Homework:

- Chapter 6, problems 1 - 4
For #4, derive the Hamilton equations of motion explicitly