Statistical Physics I - PHYS 663 - *Spring 2025*
Department of Physics - Northern Illinois University
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Homework

6 **HW**

due *2025-04-22*

Exams	final: **Thursday, April 24, 2025**, *12:00-14:00*

XXII. IDEAL FERMIONS AT LOW TEMPERATURES AND SOMMERFELD EXPANSION *[(5+9+3) PTS]*

Here you should apply the Sommerfeld expansion to calculate properties of Fermions at low temperatures (see lecture).

a) Calculate $E(T, N, V)$ up to order $\mathcal{O}(T^2)$ for a given single particle density of states (DoS) $\rho(\epsilon)$. Derive from that the result for the specific heat (in order $\mathcal{O}(T)$):

$$C_V(T, N, V) = \left(\frac{\partial E}{\partial T}\right)_{V,N} = \frac{\partial E}{\partial T} + \frac{\partial E}{\partial \mu}\left(\frac{\partial \mu}{\partial T}\right)_{V,N}$$

b) Derive an expression for the single particle DoS $\rho(\epsilon)$ in d dimensions for non-relativistic Fermions, with energy-momentum relation $\epsilon = \mathbf{p}^2/(2m)$. What is special for $d = 2$? Use this expression to calculate the energy, E, (order $\mathcal{O}(T^2)$), heat capacity, C_V, (order $\mathcal{O}(T)$), and pressure, $p(T, N, V) = -(\partial E/\partial V)_{N,T}$, (order $\mathcal{O}(T^2)$) - all for arbitrary dimension d.

c) Calculate the Fermi-energy and heat capacity for relativistic Fermions with linear energy-momentum relation, $\epsilon_{\mathbf{p}} = c|\mathbf{p}|$ (e.g., electrons in a white dwarf star) for arbitrary dimension d.

XXIII.   BOSE-EINSTEIN CONDENSATION *[(4+6+8) PTS]*

The grand canonical potential for an ideal (non-relativistic) Bose gas is given by

$$J(T, \mu, V) = T \ln(1 - z) - T \frac{V}{\lambda_\beta^3} \mathrm{Li}_{5/2}(z),$$

where $z = e^{\mu/T}$, $\lambda_\beta = h/\sqrt{2\pi mT}$, and $\mathrm{Li}_s(z)$ is the polylogarithm function.

a) Use the potential J and calculate the entropy $S(T, V, \mu) = -(\frac{\partial J}{\partial T})_{V,\mu}$ above and below the transition point of the Bose condensation. Show that S is continuous at the transition point and vanishes for $V = 0$ and $T = 0$.

b) Calculate the specific heat

$$C_V(T, V, z) = T \left(\frac{\partial S}{\partial T} \right)_{V,N}$$

above and below the transition. In the following the volume is fixed. Notice, that the partial derivative of $S(T, V, \mu)$ needs to be calculated while keeping N constant, i.e.,

$$\left(\frac{\partial S}{\partial T} \right)_{V,N} = \left(\frac{\partial S}{\partial T} \right)_z + \frac{\partial S}{\partial z} \left(\frac{\partial z}{\partial T} \right)_N .$$

The derivative $\partial z / \partial T$ is obtained by applying the operator $(\partial / \partial T)_N$ on both sides of

$$N(T, V, \mu) = - \left(\frac{\partial J}{\partial \mu} \right)_{T,V} = \frac{z}{1 - z} + \frac{V}{\lambda_\beta^3} \mathrm{Li}_{3/2}(z) . \tag{1}$$

Is the specific heat continuous at the transition point?

c) Calculate the inverse isothermal compressibility

$$K_T^{-1} = -V \left(\frac{\partial p}{\partial V} \right)_{T,N}$$

above and below the transition. Show that K_T diverges at the transition point. Notice again, that the derivative is to be calculated for constant N (T is fixed):

$$p \approx \frac{T}{\lambda_\beta^3} \mathrm{Li}_{5/2}(z) \Rightarrow \left(\frac{\partial p}{\partial V} \right)_N = \left(\frac{\partial p}{\partial V} \right)_z + \frac{\partial p}{\partial z} \left(\frac{\partial z}{\partial V} \right)_N .$$

In order to calculate $\left(\frac{\partial V}{\partial z} \right)_{T,N}$ solve eq. (1) for V. Use the above results to calculate K_T as function of the specific volume $v = V/N$ near the critical volume $v_c(T) = \lambda_\beta^3 / \mathrm{Li}_{3/2}(1)$ at which the Bose condensation occurs, i.e., for $v - v_c \ll v_c$. Use the series expansion

$$\mathrm{Li}_{3/2}(z) = \mathrm{Li}_{3/2}(1) \left(1 - 1.36\sqrt{1 - z} + \dots \right) .$$

If you calculated all correctly, S is continuous at the transition point and vanishes at $v = 0$ (or $T = 0$). Furthermore, $C_V(T, V, z)$ has a jump in the first derivative and K_T diverges at the transition point. Therefore, the Bose-Einstein condensation (BEC) is a phase transition of second order, if one considers the mixed phase of gas and condensate in the region $v < v_c$ as a new phase. On the other hand, if the new phase is only the phase when all particles are in the condensate at $T = 0$ (this state corresponds to the specific volume $v = 0$, since there is no repulsion), the BEC is a phase transition of first order for all isotherms.